# Logic, Courcelle's theorem and Applications

M. Praveen

The Institute of Mathematical Sciences, India

IMPECS School on Parameterized and Exact Computation
December 2010

# Outline

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# Monadic Second Order (MSO) logic of graphs

- Let $x, y, x_1, x_2, x_3, \ldots$ be variables that denote vertices.
- Let $X, Y, X_1, X_2, X_3, \ldots$ be variables that denote subsets of vertices.
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- Monadic Second Order logic formulas (denoted as $\phi, \phi_1, \phi_2$ etc.) are those that can be constructed using the following:
  - $x \in Y$
  - $x_1 = x_2$
  - $E(x_1, x_2)$
  - $\phi_1 \wedge \phi_2$, $\phi_1 \vee \phi_2$, $\neg\phi_1$
  - $\exists x \phi, \forall x \phi$
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  - $\exists x \phi, \forall x \phi$
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  - For Counting MSO, add $|X| \equiv q \mod p$, $p, q \in \mathbb{N}$.

# MSO logic of graphs contd…

- In $\exists x \phi$, all occurrences of $x$ inside $\phi$ are said to be bound by the quantifier $\exists$ occurring in front of $\phi$. Similarly for $\forall x \phi$, $\exists X \phi$ and $\forall X \phi$.

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- Variables not bounded by any quantifier are said be free. Ex: $\exists x_1 E(x_2, x_1)$.

# Courcelle's theorem

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- [Courcelle's theorem]: Checking whether $G$ satisfies $\phi$ is Fixed Parameter Tractable. There is an algorithm with running time $f(\mathit{treewidth}(G), \mathit{size}(\phi))n$.

# Example: CNF SAT

$$\underset{c\ell_1}{(x_1 \vee \neg x_2)} \quad \wedge \quad \underset{c\ell_2}{(x_3 \vee x_4)} \quad \wedge \quad \underset{c\ell_3}{x_5} \quad \wedge \quad \underset{c\ell_4}{(\neg x_6 \vee \neg x_7)}$$

CNF Satisfiability: There is a subset of variables such that, for every clause

Either there is a variable in the subset occurring positively

Or there is a variable not in the subset occurring negatively.

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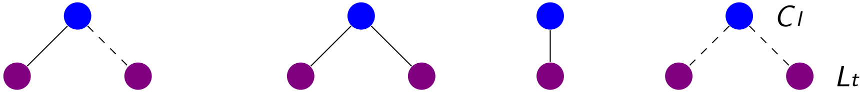


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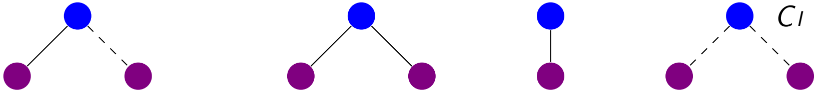


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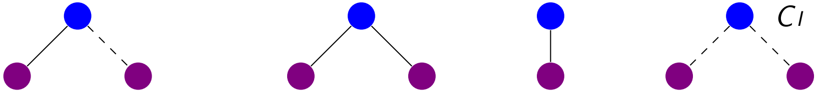


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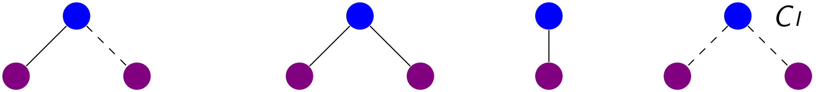


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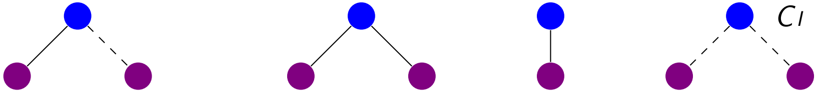


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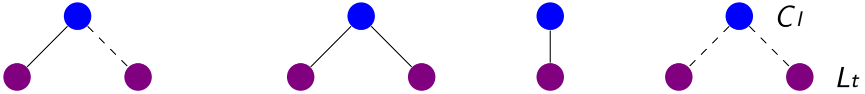


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$$\exists x_1, x_2, x_3 \quad \forall x$$
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$$(x = x_1 \vee x = x_2 \vee x = x_3) \tag{3}$$
$$\vee \exists y \quad E(x, y) \wedge (y = x_1 \vee y = x_2 \vee y = x_3)$$
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For each dominating set of size $k$, a formula can be written.

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Smallest dominating set: what is the size of a smallest subset $X$ of vertices such that $G$ satisfies $ds(X)$?

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- Many other extensions are also proved: adding conditions like $|X_1| > |X_2|$, $|X_1| + |X_2| \leq |X_3|$ and so on. However, the degree of the polynomial in the running time depends on the number of free variables.

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- The second vertex is $x$: $\binom{a}{0}\binom{a}{1}\binom{a}{0}\binom{a}{0}\binom{a}{0}$.
- The first, third and fourth vertices form the set $X$: $\begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$.

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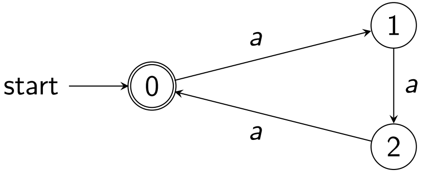

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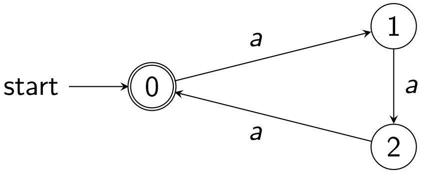
- 


- Question: For what kind of questions can we construct finite state automata?

# Answer

- Answer [Büchi, Elgot, Trakhtenbrot theorem]: For precisely those questions that can be stated in the MSO logic of path graphs.

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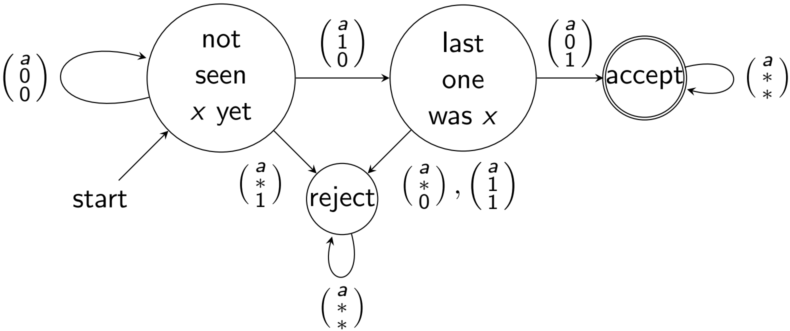
- MSO logic of path graphs: In MSO logic of graphs, replace $E(x, y)$ by $y = x + 1$.
- Automaton for checking $y = x + 1$: $\begin{pmatrix} a \\ x \\ y \end{pmatrix}$



# Constructing automaton for MSO formulas

Automaton for checking $y = x$: Exercise.

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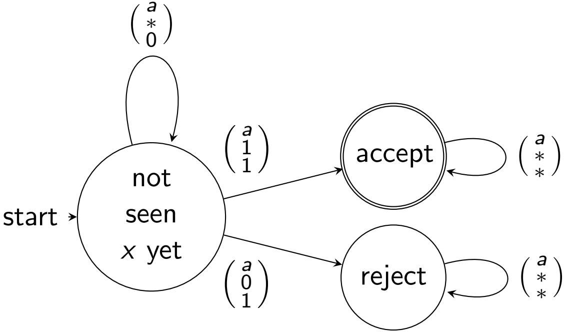
Automaton for checking $x \in X$: $\begin{pmatrix} a \\ X \\ x \end{pmatrix}$



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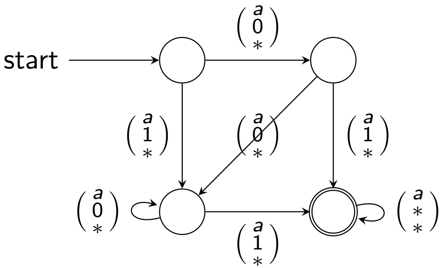
- $\exists x \phi(x, y)$: Suppose $A_{\phi(x,y)}$ is already constructed.



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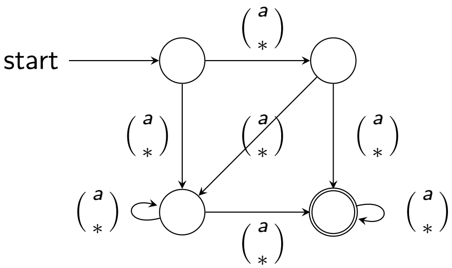

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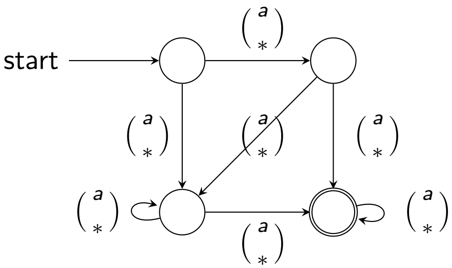


- That's all! A deterministic automaton may become non-deterministic.

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- $\forall x \phi(x)$ is same as $\neg \exists x \neg \phi(x)$.
- Construct a non-deterministic automaton for $\exists x \neg \phi(x)$ and complement it. This needs determinization and involves an exponential blow-up.

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- Fixed Parameter Tractable when $|\phi|$ is a parameter.

# Extension to graphs that are very near to being paths

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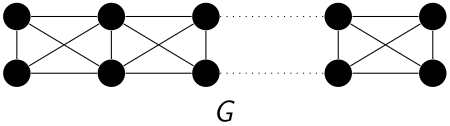


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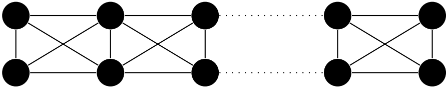

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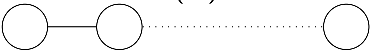

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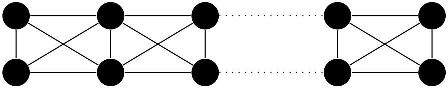

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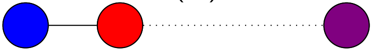

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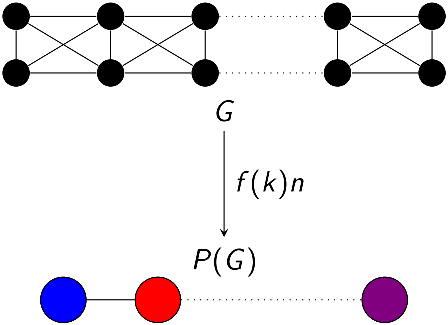


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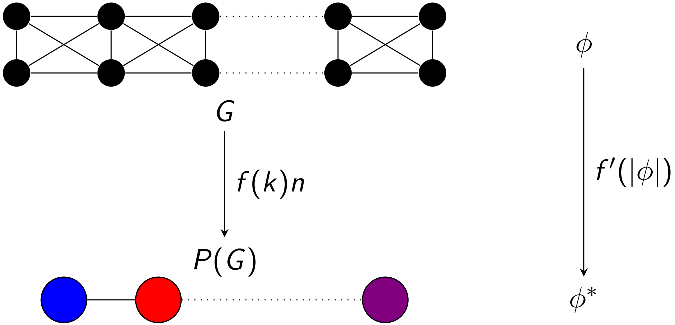

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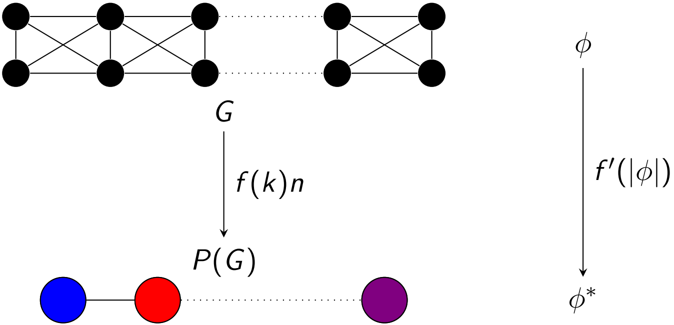


$G$ satisfies $\phi$ iff $P(G)$ satisfies $\phi^*$.

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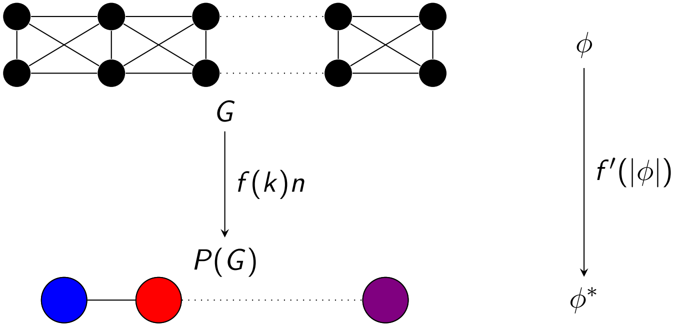


$G$ satisfies $\phi$ iff $P(G)$ satisfies $\phi^*$.
Check if $A_{\phi^*}$ accepts $P(G)$.

# Extension to treewidth

- [Doner, Thatcher, Wright]: Analogue of BET theorem for trees.

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- [Doner, Thatcher, Wright]: Analogue of BET theorem for trees.
- For checking MSO properties of graphs with bounded treewidth, use tree decomposition instead of path decomposition. Use tree automata instead of the usual string automata.

# Lower bounds

- Consider the sentence $\psi = \forall x_1 \exists x_2 \forall x_3 \exists x_4 \cdots \forall x_9 \exists x_{10} \phi(x_1, \ldots, x_{10})$.
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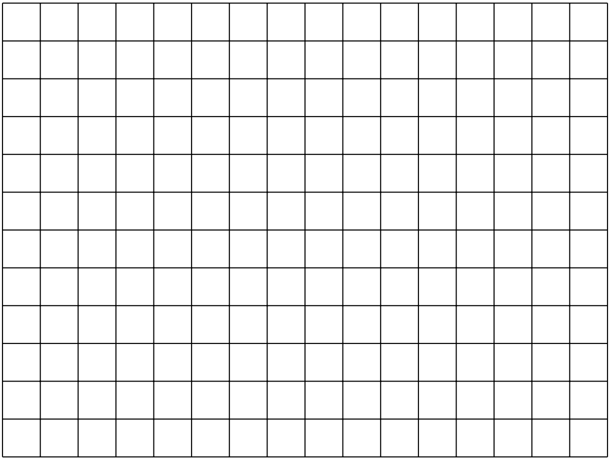
- Example: For planar graphs, $f(r) = 3r$.

# Bounded local treewidth



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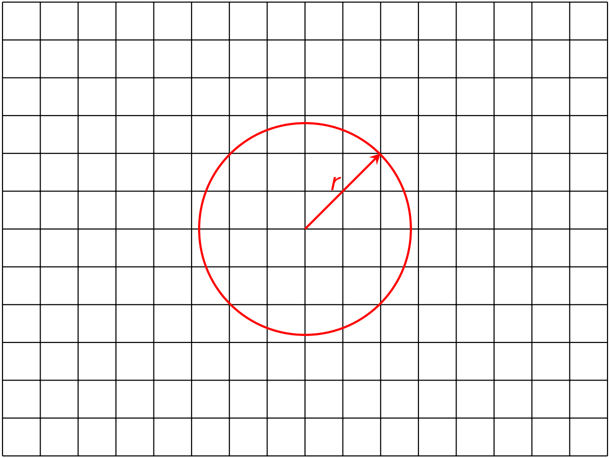

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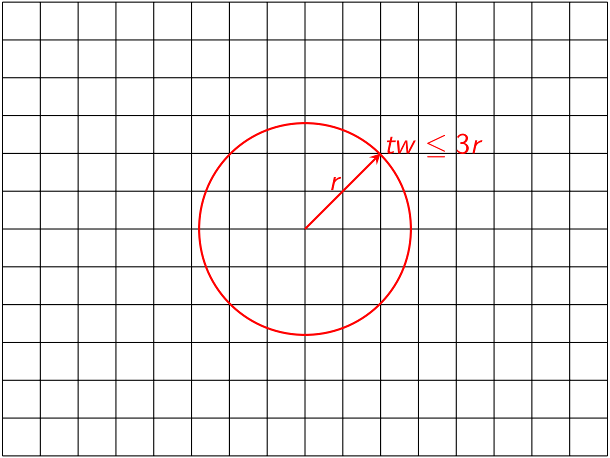

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- Proof relies on Gaifman's locality theorem: A given FO sentence can only reason about a fixed number of pairwise disjoint spheres that satisfy some FO property.

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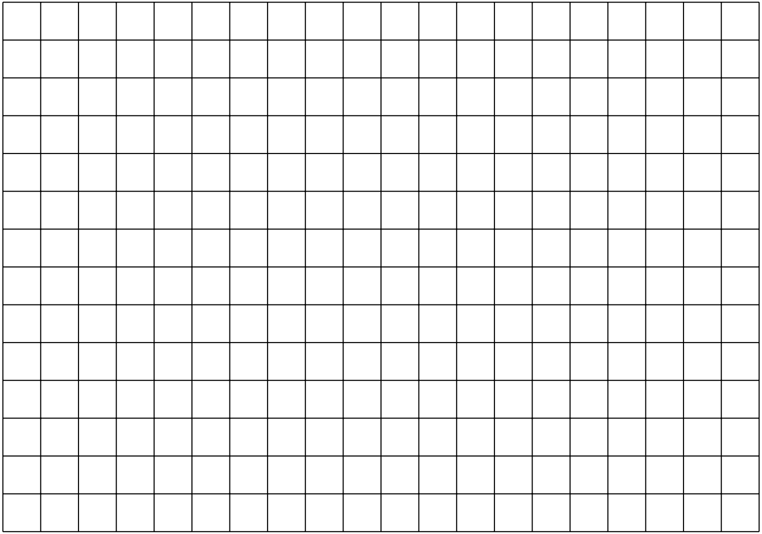

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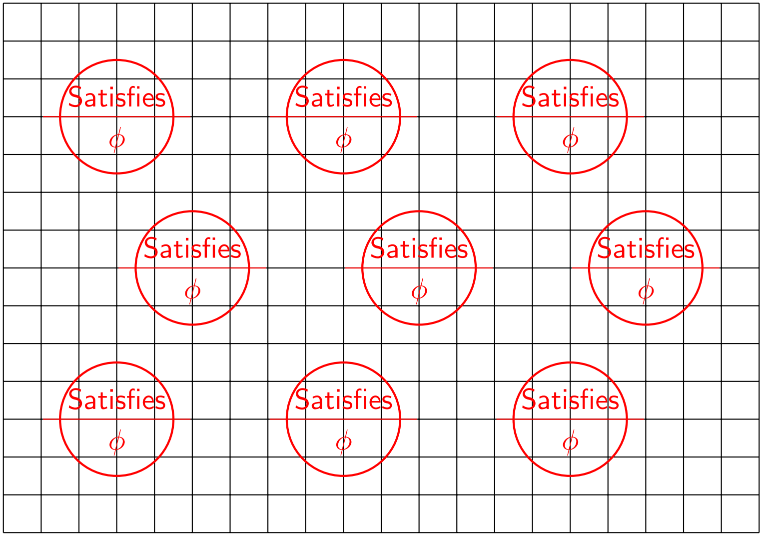

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- There are 3 equivalence classes for this particular problem. They are called Myhill-Nerode classes.

# Application to Kernelization

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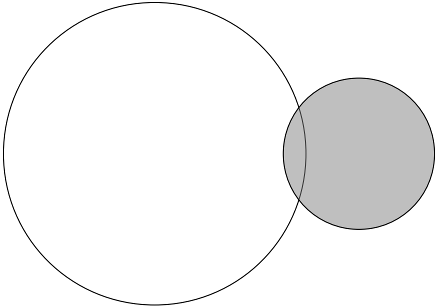

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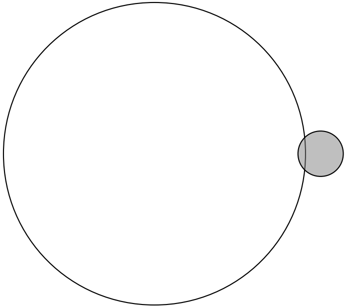
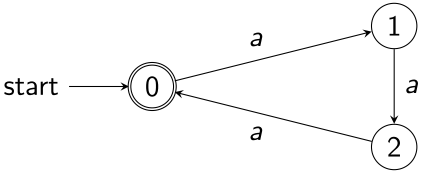


In a big enough graph, there will always be a Protrusion.
Replace by a smallest one in the same Myhill-Nerode class.

# Designing dynamic programming algorithms

- Myhill-Nerode classes have close relationship with states of a finite automaton. Example:



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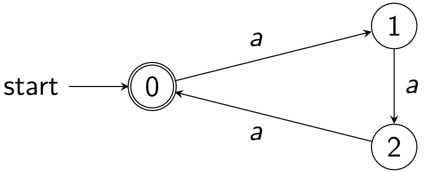


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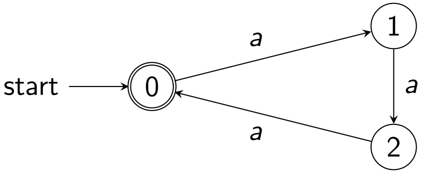


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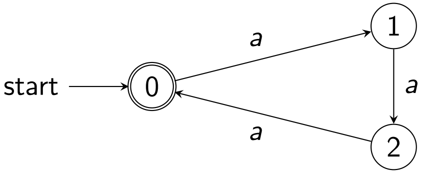


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Thank you. Questions?

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