1. Approximations of $Post^*(X_0) \cap D_0$

$P \subseteq \mathbb{Z}^d$ is periodic ~~of~~ if $\vec{0} \in P$ and $P + P \subseteq P$. Our approximation will be called linearizations $lin(P)$.

2. Property of VAS that makes above approximations well-behaved

$Post^*(X_0) \cap D_0$ is a finite union of sets of the form $b + P$, $b \in \mathbb{Z}^d$, $P \subseteq \mathbb{Z}^d$ is an ~~periodic~~ asymptotically definable periodic set. (i.e. $\mathbb{Q}_{\geq 0} P$ is definable in $FO(\mathbb{Q}, +, \leq, 0)$)

3. Our approximations are well-behaved

Linearizations of asymptotically definable periodic sets are finitely generated (and hence definable in Presburger arithmetic).

4. Taking approximations can not go on forever

$$\underset{\substack{\rotatebox{90}{$\subseteq$} \\ S}}{(Post^*(X_0) \cap D_0)} \cap \underset{\substack{\rotatebox{90}{$\subseteq$} \\ T}}{(Pre^*(Y) \cap D_0)} = \emptyset \text{ implies}$$

$$\text{dimension}(S \cap T) < \max\{\dim(Post^*(X_0) \cap D_0), \dim(Pre^*(Y) \cap D_0)\} \leq \dim(D_0)$$

Linearizations

[last para, page 4]

Definition: The topological closure of $X \subseteq \mathbb{Q}^d$ is the set $\overline{X}$ of vectors $\vec{r} \in \mathbb{Q}^d$ s.t. for every $\epsilon \in \mathbb{Q}_{>0}$ there exists $\vec{x} \in X$ satisfying $\|\vec{r} - \vec{x}\|_\infty < \epsilon$.

* Show Fig. 2 on projector *

[Beginning of section 5]

Definition: For a periodic set $P \subseteq \mathbb{Z}^d$,

$$lin(P) = (P-P) \cap \overline{\mathbb{Q}_{>0} P}$$

[Lemma 5.1]

Lemma: The linearization of an asymptotically definable periodic set is finitely generated.

Proof: Let $V = \mathbb{Q}_{>0}P - \mathbb{Q}_{>0}P$ be the vector space generated by P.

$\mathbb{Q}_{\geq 0}P \subseteq V$ & V is closed. $\therefore$ $C = \overline{\mathbb{Q}_{\geq 0}P} \subseteq V$.

* The topological closure of a definable conic set is finitely generated (to be proved next).

$\therefore$ C is finitely generated, by $\{\vec{c}_1, \cdots, \vec{c}_k\} \subseteq C$.
$\vec{c}_j \in C \subseteq V = \mathbb{Q}_{\geq 0}P - \mathbb{Q}_{\geq 0}P$, we can multiply all c_js by some natural number so that $\vec{c}_j \in P - P$.

(The above is a brutal way of throwing out non-integer vectors from $\overline{\mathbb{Q}_{\geq 0}P}$; we throw out too many. To re-introduce them, we do the following.)

$$R = \{\vec{n} \in P - P \mid \vec{n} = \sum_{j=1}^{k} \lambda_j \vec{c}_j,\ 0 \leq \lambda_j < 1\}$$

Periodic set generated by $R \cup \{\vec{c}_1, \cdots, \vec{c}_j\} = (P - P) \cap \overline{\mathbb{Q}_{\geq 0}P}$.

$\subseteq$: by construction

$\supseteq$: Let $\vec{x} \in (P-P) \cap \overline{\mathbb{Q}_{\geq 0}P}$. $\exists \mu_1, \cdots, \mu_k \in \mathbb{Q}_{\geq 0}$:
$\vec{x} = \mu_1 \vec{c}_1 + \cdots + \mu_k \vec{c}_k$. Let $\mu_j = n_j + \lambda_j$, $0 \leq \lambda_j < 1$.

$$\underset{P-P}{\vec{x}} = \underbrace{n_1 \vec{c}_1 + \cdots + n_k \vec{c}_k}_{\underset{P-P}{\in}} + \underbrace{\lambda_1 \vec{c}_1 + \cdots + \lambda_k \vec{c}_k}_{\Rightarrow\ P-P,\ \in R}$$

$\therefore$ x is generated by $R \cup \{\vec{c}_1, \cdots, \vec{c}_j\}$.

QED*

Lemma 3.7

Lemma: The topological closure of a set definable in FO($\mathbb{Q}$, +, $\leq$, 0) is a finite union of finitely generated conic sets.

Proof: Let $X \subseteq \mathbb{Q}^d$ be a ~~def~~ set definable in FO($\mathbb{Q}$, +, $\leq$, 0). Take the defining formula, eliminate quantifiers & convert to DNF to infer that

$$\cancel{X = \bigcup_{\text{finite}} \{\vec{x} \in \mathbb{Q}^d \mid \sum_{i=1}^{d} h(i)\vec{x}(i) \# 0\}, \# \in \{>, \geq\}}$$

$$X = \bigcup_{\text{finite}} X_j,$$

$$X_j = \bigcap_{\text{finite}} \{\vec{x} \in \mathbb{Q}^d \mid \sum_{i=1}^{d} h(i)\vec{x}(i) \# 0\}, \# \in \{>, \geq\}.$$

Let $R_j = \bigcap \{\vec{x} \in \mathbb{Q}^d \mid \sum_{i=1}^{d} h(i)\vec{x}(i) \geq 0\}$.

By duality, R_j is finitely generated & $R = \bigcup R_j$ is closed. We claim that $\overline{X} = R$.

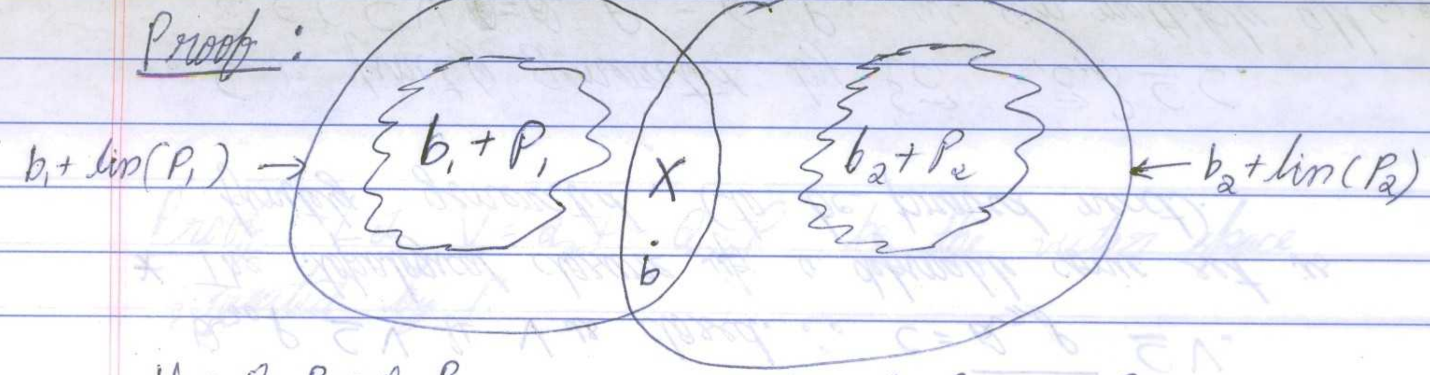
$\subseteq$: $X_j \subseteq R_j \Rightarrow X \subseteq R$. R is closed $\Rightarrow \overline{X} \subseteq R$.

$\supseteq$: Let $\vec{r} \in R_j$. $\cancel{\sum_{i=1}^{d} h(i)\vec{r}(i) = 0}$ For any $\vec{x}_j \in X_j$, $\vec{r} + \cancel{x_j}\ \mathbb{Q}_{>0}\vec{x}_j \subseteq X_j$.

QED.

Theorem 5.2 **Theorem** : $\vec{b}_1, \vec{b}_2 \in \mathbb{Z}^d$, $P_1, P_2 \subseteq \mathbb{Z}^d$ asymptotically definable periodic sets. If $(b_1 + P_1) \cap (b_2 + P_2) = \emptyset$, then $\dim[(b_1 + lin(P_1)) \cap (b_2 + lin(P_2))] < \max\{\dim(b_1+P_1), \dim(b_2+P_2)\}$

Proof :



$V_1 = \mathbb{Q}_{\geq 0} P_1 - \mathbb{Q}_{\geq 0} P_1$ $\qquad$ $V_2 = \mathbb{Q}_{\geq 0} P_2 - \mathbb{Q}_{\geq 0} P_2$

$C_1 = \overline{\mathbb{Q}_{\geq 0} P_1}$ $\qquad$ $C = C_1 \cap C_2$ $\qquad$ $C_2 = \overline{\mathbb{Q}_{\geq 0} P_2}$

C_1 & C_2 are finitely generated.

$\dim(b_j + P_j) = rank(V_j)$ (to be proved later).

$X \subseteq b + V_1$, $X \subseteq b + V_2$

$\Rightarrow X \subseteq b + (V_1 \cap V_2)$

$= b + V$.

If $V \subsetneq V_1$, then $\dim(X) \leq rank(V) < rank(V_1)$ and we are done. Similarly, if $V \subsetneq V_2$, we are done. The remaining case is $V_1 = V_2 = V$.

Claim: $C \subsetneq V$.

Proof : Suppose not: $C = V$. $int(C) = \{c \in C \mid \exists \epsilon \in \mathbb{Q}_{>0} :$ $c + v \in C \ \forall v \in V$ where $\|v\|_\infty \leq \epsilon\}$

Lemma 3.5

$\text{int}(C) = \text{int}(C_1) \cap \text{int}(C_2)$. [to be proved later]

$= \text{int}(Q_{\geq 0} P_1) \cap \text{int}(Q_{\geq 0} P_2)$

$= \text{int}(Q_{\geq 0}(P_1 \cap P_2))$ [to be proved later]

$\therefore \exists\, c \in \text{int}(C) \cap P_1 \cap P_2$

$b \in b_1 + \text{lin}(P_1)$

$\Rightarrow b = b_1 + (p_1 - p_1')$, $p_1, p_1' \in P_1$ ($\because -p_1'$ is in the VS generated by C)

$\exists\, n_1 \in \mathbb{N}$ large enough s.t. $n_1 c + (-p_1') \in C_1$

$\therefore \exists\, n_1' \in \mathbb{N}$ large enough s.t. $n_1' n_1 c - n_1' p_1' \in P_1$

$\therefore n_1' n_1 c - p_1' \in (n_1' - 1) p_1' + P_1 \subseteq P_1$

$\therefore b_1 + p_1 + n_1' n_1 c - p_1' \in b_1 + p_1 + P_1$

$\Rightarrow b + k_1 c \in b_1 + P_1$

lly $b + k_2 c \in b_2 + P_2$

$\therefore b + (k_1 + k_2) c \in (b_1 + P_1) \cap b_2 + P_2$, contradiction.

Since $C \subsetneq V$, $\exists \vec{h} \in V \setminus \{0\}$ [to be proved later] such that

$$C_1 \subseteq \{v \in V \mid \sum_{i=1}^{d} h(i)\, v(i) \geq 0\}$$

$$C_2 \subseteq \{v \in V \mid \sum_{i=1}^{d} h(i)\, v(i) \leq 0\}.$$

Multiply by a number large enough so that $h \in \mathbb{Z}^d$.

$\vec{x} \in X$.

$\vec{x} - \vec{b}_1 \in C_1 \Rightarrow \sum_{i=1}^{d} \vec{h}(i)(\vec{x}(i) - \vec{b}_1(i)) \geq 0$

$\vec{x} - \vec{b}_2 \in C_2 \Rightarrow \sum_{i=1}^{d} \vec{h}(i)(\vec{x}(i) - \vec{b}_2(i)) \leq 0.$

Let $z_1 = \sum_{i=1}^{d} \vec{h}(i)\, \vec{b}_1(i)$, $z_2 = \sum_{i=1}^{d} \vec{h}(i)\, \vec{b}_2(i)$.

$X = \bigcup_{z=z_1}^{z_2} X_z$, $X_z = \{x \in X \mid \sum_{i=1}^{d} \vec{h}(i)\, \vec{x}(i) = z\}$.

For $\vec{x}_z \in X_z$, $X_z \subseteq \vec{x}_z + W$, $W = \{v \in V \mid \sum_{i=1}^{d} \vec{h}(i)\, \vec{v}(i) = 0\}$.

$\vec{h} \in V \setminus W$ ($\vec{h}$ is not $\vec{0}$).

$\therefore W \subsetneq V \Rightarrow \text{rank}(W) < \text{rank}(V) \Rightarrow \dim(X_z) < \text{rank}(V)$

QED*

Lemma 5.4

Lemma 5.4: Let $V = Q_{\geq 0}P - Q_{\geq 0}P$. $rank(V) = dim(P)$.

Proof: Claim: $P \subseteq \bigcup_{j=1}^{k} V_j \Rightarrow \exists j \in \{1, \dots, k\}: P \subseteq V_j$.

Proof: By induction on k. $k=1$ is immediate.
Suppose $P \subseteq \bigcup_{j=1}^{k+1} V_j$. If $P \subseteq V_{k+1}$, we are done.
~~o/w, $p \in P \setminus V_{k+1}$. We will show~~

Now, we will show $P \subseteq \bigcup_{j=1}^{k} V_j$. Let $\vec{x} \in P$. If

$x \notin V_{k+1}$, we are done. So suppose $\vec{x} \in V_{k+1}$. $\exists \vec{p} \in P \setminus V_{k+1}$

Since $p + nx \in P$ $\forall n$, $\exists n < n'$: $\vec{p} + n\vec{x}, \vec{p} + n'\vec{x} \in V_j$

$\therefore n'(\vec{p} + n\vec{x}) - n(\vec{p} + n'\vec{x}) \in V_j$

$\Rightarrow p \in V_j$, $j \neq k+1$

$(\vec{p} + n'\vec{x}) - (\vec{p} + n\vec{x}) \in V_j$

$\Rightarrow x \in V_j$

$\Rightarrow P \subseteq \bigcup_{j=1}^{k} V_j$. By IH, $P \subseteq V_j$ for some j.

$\square$

$\dim(P) \leq \operatorname{rank}(V)$.

$P \subseteq \bigcup_{j=1}^{k} b_j + V_j$, $\operatorname{rank}(V_j) \leq \dim(P)$ $\forall j \in \{1, \ldots, k\}$

Let $J = \{ j \in \{1, \ldots, k\} \mid b_j \in V_j \}$. We will show that $P \subseteq \bigcup_{j \in J} V_j$.

Since $n\vec{p} \in P$ $\forall n$, $\exists n < n'$: $n\vec{p}, n'\vec{p} \in b_j + V_j$

$\Rightarrow n'\vec{p} - n\vec{p} \in V_j$ $\qquad \Rightarrow b_j \in n\vec{p} - V_j \subseteq V_j$

$\Rightarrow \vec{p} \in V_j$ $\qquad \Rightarrow j \in J$.

By claim, $\exists j \in J$: $P \subseteq V_j$. Since V is the vector space generated by P, $V \subseteq V_j$. $\therefore \operatorname{rank}(V) \leq \operatorname{rank}(V_j) \leq \dim(P)$.

QED.

$\boxed{\text{Lemma 4.5}}$

Lemma: $(\mathbb{Q}_{\geq 0} P_1) \cap (\mathbb{Q}_{\geq 0} P_2) = \mathbb{Q}_{\geq 0} (P_1 \cap P_2)$

Proof: $P_1 \cap P_2 \subseteq (\mathbb{Q}_{\geq 0} P_1) \cap (\mathbb{Q}_{\geq 0} P_2)$

$\Rightarrow \mathbb{Q}_{\geq 0}(P_1 \cap P_2) \subseteq (\mathbb{Q}_{\geq 0} P_1) \cap (\mathbb{Q}_{\geq 0} P_2)$

Let $c \in (\mathbb{Q}_{\geq 0} P_1) \cap (\mathbb{Q}_{\geq 0} P_2)$

$\exists n_1, n_2$: $n_1 c \in P_1$, $n_2 c \in P_2$

$n_1 n_2 c \in P_1 \cap P_2 \Rightarrow c \in \mathbb{Q}_{\geq 0}(P_1 \cap P_2)$ $\qquad$ QED.

Lemma 5.5

Lemma: Let $C_{\leq}$ and $C_{\geq}$ be finitely generated conic sets generating the same vector space V. If the vector space generated by $C_{\geq} \cap C_{\leq}$ is strictly included in V, there exists a vector $\vec{h} \in V \setminus \{0\}$ s.t. $\forall \# \in \{\leq, \geq\}$,

$$C_{\#} \subseteq \{\vec{v} \in V \mid \sum_{i=1}^{d} \vec{h}(i)\vec{v}(i) \# 0\}.$$

Proof: From Lemma 3.5, $\exists$ finite $H_{\leq}, H_{\geq} \subseteq_f V \setminus \{0\}$:

$$C_{\#} = \bigcap_{\vec{h} \in H_{\#}} \{\vec{v} \in V \mid \sum_{i=1}^{d} \vec{h}(i)\vec{v}(i) \geq 0\}.$$

$$int(C_{\#}) = \bigcap_{\vec{h} \in H_{\#}} \{\vec{v} \in V \mid \sum_{i=1}^{d} \vec{h}(i)\vec{v}(i) > 0\}.$$

If $\exists \vec{c} \in int(C_{\geq}) \cap int(C_{\leq})$, $\exists \varepsilon \in \mathbb{Q}_{>0}$: $\lambda c + v \in C_{\leq} \cap C_{\geq}$ for $\vec{v} \in V$ s.t. $\|\vec{v}\|_{\infty} < \varepsilon$. This will imply that the vector space generated by $C_{\geq} \cap C_{\leq}$ contains V, a contradiction. Hence, $int(C_{\geq}) \cap int(C_{\leq})$ is empty. $H = H_{\leq} \cup H_{\geq}$

$$\bigcap_{h \in H} \{\vec{v} \in V \mid \sum_{i=1}^{d} \vec{h}(i)\,\vec{v}(i) > 0\} = \phi.$$

From Farkas's lemma, $\exists$ non-zero $f: H \to \mathbb{Q}_{\geq 0}$: $\sum_{\vec{h} \in H} f(\vec{h})\,\vec{h} = \vec{0}$. Let $\vec{a} = \sum_{\vec{h} \in H_{\geq}} f(\vec{h})\,\vec{h}$, $\vec{b} = \sum_{\vec{h} \in H_{\leq}} f(\vec{h})\vec{h}$.

If $\vec{a} = \vec{0}$, then $\vec{b} = \vec{0}$. Pick a $\vec{h} \in H$ s.t. $f(\vec{h}) \neq 0$. If $h \in H_{\geq}$, then $\vec{a} = \vec{0}$ implies $int(C_{\geq}) = \phi$ and if $h \in H_{\leq}$, then $\vec{b} = 0$ implies $int(C_{\geq}) = \phi$, both contradictions. Hence $\vec{a} \neq \vec{0}$.

Now, $\forall \vec{c} \in C_{\leq}, \quad \sum_{i=1}^{d} \vec{a}(i)\, \vec{c}(i) \geq 0.$

$\forall \vec{c} \in C_{\leq}, \quad \sum_{i=1}^{d} \vec{b}(i)\, \vec{c}(i) \geq 0$

$\vec{a}+\vec{b}=\vec{0} \Rightarrow \forall \vec{c} \in C_{\leq}, \quad \sum_{i=1}^{d} \vec{a}(i)\, \vec{c}(i) \leq 0.$

QED.

---

* Start here

Vector Addition Systems

Main Result: $Post^*(X_0) \cap D_0$ is a finite union of sets of the form $b+P$, $P \subseteq \mathbb{Z}^d$ is a periodic set s.t. $\mathbb{Q}_{\geq 0} P$ is definable in $FO(\mathbb{Q}_{\geq 0}, \leq, +, 0, 1)$, provided X_0 and D_0 are Presburger definable.

Lemma 6.2 To conclude that $Post^*(X_0) \cap D_0$ is a finite union of ----, it is enough to prove that $\xrightarrow{*} \cap D$ is a finite union of ... for every presburger set $D \subseteq \mathbb{Z}^{2d}$.

Proof:

$X_0 \times D_0$ is a Presburger set.

$\therefore \xrightarrow{*} \cap (X_0 \times D_0) = \bigcup_{j=1}^{k} (a_j, b_j) + R_j$, $\mathbb{Q}_{\geq 0} R_j$ is definable.

$Post^*(X_0) \cap D_0 = \bigcup_{j=1}^{k} b_j + P_j$, $P_j = \{\vec{v} \in \mathbb{Z}^d \mid \exists (\vec{u}, \vec{v}) \in R_j\}$.

$R_j + R_j \subseteq R_j \Rightarrow P_j + P_j \subseteq P_j$.

$\mathbb{Q}_{\geq 0} R_j$ definable $\Rightarrow C_j = \{\vec{v} \in \mathbb{Q}^d \mid \exists (\vec{u}, \vec{v}) \in \mathbb{Q}^d_{\geq 0} R_j\}$ is definable.

We will show that $\mathbb{Q}_{\geq 0} P_j = C_j$.

$\subseteq$: $P_j \subseteq C_j$ & $\mathbb{Q}_{\geq 0} C_j \subseteq C_j \Rightarrow \mathbb{Q}_{\geq 0} P_j \subseteq C_j$

$\supseteq$: $\vec{v} \in C_j \Rightarrow \exists \vec{u} \in \mathbb{Q}^d : (\vec{u}, \vec{v}) \in \mathbb{Q}^d_{\geq 0} R_j \Rightarrow \exists \lambda : (\vec{u}, \vec{v}) \in \lambda R_j$

$\Rightarrow \exists n : (n\vec{u}, n\vec{v}) \in n\lambda R_j \subseteq R_j \Rightarrow n\vec{v} \in P_j \Rightarrow \vec{v} \in \mathbb{Q}_{\geq 0} P_j$.

QED.

$\vec{m} \in N^d$, $\vec{n} \xrightarrow{*}_{m} \vec{s}$ iff $\vec{m} + \vec{n} \xrightarrow{*} \vec{m} + \vec{s}$

(show fig. 8)

$\rho = m_0 \cdots m_k$, $\xrightarrow{*}_{\rho} = \xrightarrow{*}_{m_0} \circ \cdots \circ \xrightarrow{*}_{m_k}$

Lemma 9.2: It is enough to show that $\xrightarrow{*}_{\rho} \cap P$ is ~~a finite union of~~ asymptotically definable for every finitely generated periodic set P.

Proof: $\vec{p}, \vec{p'} \in P$. $\vec{p} \leq_P \vec{p'} \xleftrightarrow{def} p' \in p + P$.

P is finitely generated $\Rightarrow$ $\leq_P$ is a well-order.

$\vec{p}_1 \quad \vec{p}_2 \quad \vec{p}_3 \quad \cdots$

$\vec{g}_1 \quad n_1' \quad n_2' \quad n_3'$

$\vdots \qquad \vdots \qquad \vdots$

$\vec{g}_k \quad n_1^k \quad n_2^k \quad n_3^k$

$\Omega_{\vec{m}, \rho, \vec{n}} = \{ \rho \mid (src(\rho), tgt(\rho)) \in (\vec{m}, \vec{n}) + P \}$.

$\rho \preceq_P \rho' \xleftrightarrow{def} \rho \preceq \rho'$ and $(src(\rho), tgt(\rho)) - (\vec{m}, \vec{n}) \leq_P$

$(src(\rho'), tgt(\rho')) - (\vec{m}, \vec{n})$.

~~$\lesssim_p$~~ $\lesssim_p$ is a well-order [to be proven later]

$\therefore$ $\min_{\lesssim_p}(\Omega_{\vec{m},P,\vec{n}})$ is finite.

we will show that $\xrightarrow{*} \cap ((\vec{m},\vec{n})+P) =$

$$\bigcup_{\rho \in \min_{\lesssim_p}(\Omega_{\vec{m},P,\vec{n}})} (src(\rho), tgt(\rho)) + (\xrightarrow{*}_\rho \cap P).$$

$\supseteq$: $(src(\rho), tgt(\rho)) + \xrightarrow{*}_\rho \subseteq \xrightarrow{*}$ [to be proved later]

$\Rightarrow$ Since $(src(\rho), tgt(\rho)) \in (\vec{m},\vec{n}) + P$, we are done.

$\subseteq$: Suppose $(\vec{x},\vec{y}) \in \xrightarrow{*} \cap ((\vec{m},\vec{n}) + P)$. There is a run $\rho' \in \Omega_{\vec{m},P,\vec{n}}$. $\vec{x} \xrightarrow{\rho'} \vec{y}$. $\exists$ $\rho \in \min_{\lesssim_p}(\Omega_{\vec{m},P,\vec{n}})$

s.t. $\rho \lesssim_p \rho'$. $\Rightarrow$ $\rho \lesssim \rho'$

$\Rightarrow (\vec{x},\vec{y}) \in (src(\rho), tgt(\rho)) + \xrightarrow{*}_\rho$

[to be proved later].

$(src(\rho), tgt(\rho)) - (\vec{m},\vec{n}) \leq_p$
$(src(\rho'), tgt(\rho')) - (\vec{m},\vec{n})$ $\Rightarrow (\vec{x},\vec{y}) \in (src(\rho), tgt(\rho)) + (\xrightarrow{*}_\rho \cap P)$.
$(x', y') - (\vec{m},\vec{n})$

QED.

Lemma: $(src(\rho), tgt(\rho)) + \xrightarrow{*}_{\rho} \subseteq \xrightarrow{*}$

Proof: Let $\rho = m_1 m_2$. To prove $(m_1, m_2) + \xrightarrow{*}_{m_1 m_2} \subseteq \xrightarrow{*}$

In word, if ~~$s + m_1 \cdot (s, t) \longrightarrow$~~ $s \xrightarrow{*}_{m_1 m_2} t$, then $s + m_1 \xrightarrow{*} t + m_2$.

$s \xrightarrow{*}_{m_1 m_2} t \Rightarrow \exists m : \; s \xrightarrow[m_1]{*} m \xrightarrow[m_2]{*} t$

$$s + m_1 \xrightarrow{*} m + m_1 , \quad m + m_2 \xrightarrow{*} t + m_2$$

Since $\rho = m_1 m_2$ is a run, $m_1 \longrightarrow m_2 \Rightarrow m + m_1 \longrightarrow m + m_2$

$$s + m_1 \xrightarrow{*} m + m_1 \longrightarrow m + m_2 \xrightarrow{*} t + m_2 \quad \text{QED.}$$

$\rho \preceq \rho' \overset{def}{\Longleftrightarrow} (src(\rho'), tgt(\rho')) + \xrightarrow{*}_{\rho'} \subseteq (src(\rho), tgt(\rho)) + \xrightarrow{*}_{\rho}$

$\rho = m_0 \cdots m_k$, $\alpha(\rho) = (a_1, m_1) \cdots (a_k, m_k)$
where $a_j = m_j - m_{j-1}$.

$(a,m) \sqsubseteq (a', m') \overset{def}{\Longleftrightarrow} a = a'$ and $m \leq m'$.
$\sqsubseteq$ is a well-order. From Higman's lemma $\sqsubseteq^*$ is a well order.
$\rho \trianglelefteq \rho' \overset{def}{\Longleftrightarrow} \alpha(\rho) \sqsubseteq^* \alpha(\rho')$, $src(\rho) \leq src(\rho')$ and $tgt(\rho) \leq tgt(\rho')$.

Lemma 7.6 <u>Lemma</u>: $\rho = m_0 \cdots m_k$ and ρ' be another run.
$\rho \trianglelefteq \rho'$ iff $\exists (v_j)_{0 \leq j \leq k+1}$ of vectors in $\mathbb{N}^d$ s.t.
$\rho' = \rho_0' \cdots \rho_k'$ where $m_j + v_j \xrightarrow{\rho_j'} m_j + v_{j+1}$.
<u>Proof</u>: Let $a_j = m_j - m_{j-1}$.
($\Rightarrow$) $\alpha(\rho) \sqsubseteq^* \alpha(\rho') \Rightarrow \alpha(\rho') = w_0 (a_1, m_1') w_1 \cdots (a_k, m_k') w_k$
where $m_j' \geq m_j$. Let $v_0 = src(\rho') - src(\rho)$, $v_{k+1} = tgt(\rho') - tgt(\rho)$ and $v_j = m_j' - m_j$. $\rho = \rho_0' \cdots \rho_k'$ where ρ_j' is the run from $m_j + v_j$ to $m_j + v_{j+1}$ s.t. $\alpha(\rho_j') = w_j$.

$\rho: m_0 \xrightarrow{a_1} m_1 \qquad \xrightarrow{a_k} m_k$

~~QED~~

$\rho': \rightsquigarrow m_0 + v_0 \xrightarrow{a_1} m_1 + v_1 \quad \cdots \quad \underset{m_{k-1}+v_{k-1}}{\rightsquigarrow} m_k + v_k \xrightarrow{a_k} m_k + v_k$

$\rho: m_0 \xrightarrow{a_1} m_1 \xrightarrow{a_2} m_2 \rightarrow \cdots \xrightarrow{a_k} m_k$

$\rho': \underset{m_0 + v_0}{\rightsquigarrow} ~~m_1~~ + v_1 \xrightarrow{a_1} m_1 + v_1 \rightsquigarrow m_1 + v_2 \xrightarrow{a_2} m_2 + v_2 \rightsquigarrow \cdots \rightsquigarrow m_{k-1} + v_k \xrightarrow{a} m_k + v_k$

QED

$\boxed{\text{Lemma 7.7}}$ **Lemma**: $\rho \trianglelefteq \rho' \Rightarrow \rho \preceq \rho'$

Proof:

$$m_0 \xrightarrow{a_1} m_1 \xrightarrow{a_2} m_2 \rightarrow \cdots \xrightarrow{a_k} m_k$$

$$m_0+v_0 \overset{\rho_0'}{\rightsquigarrow} m_0+v_1 \xrightarrow{a_1} m_1+v_1 \overset{\rho_1'}{\rightsquigarrow} m_1+v_2 \xrightarrow{a_2} m_2+v_2 \quad m \rightsquigarrow m_{k-1}+v_k \xrightarrow{a_k} m_k+v_k \overset{\rho_k'}{\rightsquigarrow} m_k+v_{k+1}$$

$(m_0+v_0, m_0+v_1) + \xrightarrow{*}_{\rho_0'} \subseteq \xrightarrow{*}$

$(v_0, v_1) + \xrightarrow{*}_{\rho_0'} \subseteq \xrightarrow{*}_{m_0}$

$(v_1, v_2) + \xrightarrow{*}_{\rho_1'} \subseteq \xrightarrow{*}_{m_1}$

$(v_0, v_2) + \xrightarrow{*}_{\rho_0'\rho_1'} \subseteq \xrightarrow{*}_{m_0 m_1}$

$(v_0, v_{k+1}) + \xrightarrow{*}_{\rho'} \subseteq \xrightarrow{*}_{\rho}$

$(m_0+v_0, m_k+v_{k+1}) + \xrightarrow{*}_{\rho'} \subseteq (m_0, m_k) + \xrightarrow{*}_{\rho}$

QED.

This completes the proof that $\preceq_P$ is a wqo.

[Theorem 8.1] Lemma : $\xrightarrow{*}_P \cap P$ is asymptotically definable for every finitely generated periodic set P.

Proof : P is asymptotically definable.

[Theorem 8.1] $\xrightarrow{*}_P$ is asymptotically definable [to be proved later]

[Lemma 4.5] Asymptotic definability is closed under intersection [to be proved later].

QED.

Lemma : Asymptotic definability is closed under intersection for Periodic sets.

Proof : $(Q_{\geq 0} P_1) \cap (Q_{\geq 0} P_2) = Q_{\geq 0}(P_1 \cap P_2)$ for all periodic sets. (already proved).

$(\supseteq)$: $P_1 \subseteq Q_{\geq 0} P_1$, $P_2 \subseteq Q_{\geq 0} P_2$

$\Rightarrow P_1 \cap P_2 \subseteq (Q_{\geq 0} P_1) \cap (Q_{\geq 0} P_2)$

$\Rightarrow Q_{\geq 0}(P_1 \cap P_2) \subseteq (Q_{\geq 0} P_1) \cap Q_{\geq 0} P_2$

$(\subseteq)$: $\cancel{\vec{c} \in Q_{\geq 0}(P_1 \cap P_2)}$ $\vec{c} \in (Q_{\geq 0} P_1) \cap (Q_{\geq 0} P_2)$

$\Rightarrow \vec{c} \in \lambda_1 P_1$, $\vec{c} \in \lambda_2 P_2$

$\Rightarrow n_1 \vec{c} \in P_1$, $n_2 \vec{c} \in P_2$

$\Rightarrow n_1 n_2 \vec{c} \in P_1 \cap P_2$

$\Rightarrow c \in Q_{\geq 0}(P_1 \cap P_2)$

QED.

[Theorem 8.1] Lemma : $\xrightarrow{*}_P$ is asymptotically definable.

Proof : ~~It~~ $P = m_0 \cdots m_k$

It is enough to show $\xrightarrow{*}_m$ is asymptotically definable [to be proved later] [Lemma 8.2]

[End of Section 8] $\xrightarrow{*}_m$ is asymptotically definable [t b p l].

QED

Lemma 8.2

Lemma: It is enough to show $\xrightarrow{*}_m$ is a d

Let $R_1, R_2 \subseteq \mathbb{Z}^{d}$ be periodic relations over $\mathbb{Z}^d$.

$\mathbb{Q}_{\geq 0}(R_1 \circ R_2) = (\mathbb{Q}_{\geq 0} R_1) \circ (\mathbb{Q}_{\geq 0} R_2)$

Proof: $(\subseteq)$: $R_1 \circ R_2 \subseteq (\mathbb{Q}_{\geq 0} R_1) \circ (\mathbb{Q}_{\geq 0} R_2)$

$\mathbb{Q}_{\geq 0}(R_1 \circ R_2) \subseteq (\mathbb{Q}_{\geq 0} R_1) \circ (\mathbb{Q}_{\geq 0} R_2)$.

$(\supseteq)$: $(x, z) \subseteq (\mathbb{Q}_{\geq 0} R_1) \circ (\mathbb{Q}_{\geq 0} R_2)$

$\Rightarrow \exists y \quad (x, y) \in (\mathbb{Q}_{\geq 0} R_1), (y, z) \in (\mathbb{Q}_{\geq 0} R_2)$.

$(x, z) \in \mathbb{Q}_{\geq 0}(R_1 \circ R_2)$. $\quad (x, y) \in \lambda_1 R_1, (y, z) \in \lambda_2 R_2$

$\Uparrow \quad (n_1 x, n_1 y) \in R_1, \ (n_2 y, n_2 z) \in R_2$

$(n_1 n_2 x, n_1 n_2 z) \in R_1 \circ R_2 \Leftarrow (n_1 n_2 x, n_1 n_2 y) \in R_1, \ (n_2 n_1 y, n_2 n_1 z) \in R_2$ QED.

Lemma: $\xrightarrow{*}_m$ is asymptotically definable

Proof: Enough to prove that $\mathbb{Q}_{\geq 0} \xrightarrow{*}_m \cap V$ is finitely generated for every vector space $V \subseteq \mathbb{Q}^d$. [t b p l]

Lemma 3.8

Lemma 8.6 $\mathbb{Q}_{\geq 0} \xrightarrow{*}_m \cap V$ is finitely generated [t b p l].

QED

THEOREM 3.8: A conic set $C \subseteq \mathbb{Q}^d$ is definable iff the conic set ~~$\overline{C}$~~ $\overline{C \cap V}$ is finitely generated for every vector space $V \subseteq \mathbb{Q}^d$.

Proof: $(\Rightarrow)$: Let $X = C \cap V$.

$$X \text{ definable} \Rightarrow \overline{X} = \bigcup_{j=1}^{k} C_j, \quad C_j \text{ finitely generated.}$$

$$\vec{0} \in C_j \Rightarrow \overline{X} \subseteq \sum_{j=1}^{k} C_j.$$

$$\mathbb{Q}_{\geq 0}\overline{X} + \mathbb{Q}_{\geq 0}\overline{X} \subseteq \mathbb{Q}_{\geq 0}\overline{X} \Rightarrow \sum_{j=1}^{k} C_j \subseteq \overline{X}$$

$$\therefore \overline{X} = \sum_{j=1}^{k} C_j, \ \therefore \overline{X} \text{ is finitely generated.}$$

$(\Leftarrow)$ By induction on $n = \text{rank}(C-C)$. Base case $n=0$: $C = \{\vec{0}\}$.

Induction step: $\text{rank}(C-C) = n+1$. Let $W = C - C$.

$\overline{C} = \overline{C \cap \mathbb{Q}^d} \Rightarrow \overline{C}$ is finitely generated.

Duality $\Rightarrow$ $$\overline{C} = \bigcap_{j=1}^{K} \{\vec{x} \in W \mid \sum_{i=1}^{d} \vec{h}_j(i)\,\vec{x}(i) \geq 0\}.$$

$$\text{int}(C) = \text{int}(\overline{C}) = \bigcap_{j=1}^{K} \{\vec{x} \in W \mid \sum_{i=1}^{d} \vec{h}_j(i)\,\vec{x}(i) > 0\}.$$

$\therefore$ $\text{int}(C)$ is definable in $FO(\mathbb{Q}, +, \leq, 0, 1)$.

Since ~~$\in$~~ $\text{int}(C) \subseteq C \subseteq \overline{C}$,

$$C = \text{int}(C) \cup \bigcup_{j=1}^{K} (C \cap W_j)$$

$$W_j = \{\vec{w} \in W \mid \sum_{i=1}^{d} \vec{h}_j(i)\,\vec{w}(i) = 0\}$$

$\vec{h}_j \in W \setminus W_j$ $\therefore$ $\mathrm{rank}(W_j) < \mathrm{rank}(W) = n+1$.

$\therefore$ $C_j = C \cap W_j$ is s.t. $\mathrm{rank}(C_j - C_j) \leq \mathrm{rank}(W_j) \leq n$;

$\overline{C_j \cap V} = \overline{C \cap W_j \cap V}$ is finitely generated for every vector space V. $\therefore$ C is definable. QED.

End of section 8

~~Lemma 8.6~~ **Theorem**: $\overline{(Q_{\geq 0} \xrightarrow{*}_{m}) \cap V}$ is finitely generated for every ~~set~~ vector space.

Proof: Let $\xrightarrow{*}_{m,V} = \xrightarrow{*}_{m} \cap V$. $(Q_{\geq 0} \xrightarrow{*}_{m}) \cap V = (Q_{\geq 0} \xrightarrow{*}_{m,V})$. ~~It is enough to show that $\overline{Q_{\geq 0} P}$ is finitely generated for every Presburger periodic set P provided $\xrightarrow{*}_{m,V}$~~ QED

Lemma 8.5 $\overline{Q_{\geq 0} \xrightarrow{*}_{m,V}} = \overline{Q_{\geq 0} R_{m,V}}$

$R_{m,V}$ is a periodic Presburger set Lemma 8.4

$\overline{Q_{\geq 0} P}$ is finitely generated for every periodic PS P QED

$\Omega_{m,V} = \{ \rho \mid (src(\rho), tgt(\rho)) - (\vec{m}, \vec{m}) \in (\mathbb{N}^d \times \mathbb{N}^d) \cap V \}$.

$(r, s) \in \mathbb{N}^d \times \mathbb{N}^d \in r \xrightarrow{*}_{m,V} s \xLeftrightarrow{def}$

$\exists \rho \in \Omega_{m,V} : m + r \xrightarrow{\rho} m + s$.

$Q_{m,V} = \{ q \in \mathbb{N}^d \mid q$ occurs in some run of $\Omega_{m,V} \}$.

$I_{m,V} = \{ i \in \{1, \dots, d\} \mid \{ q(i) \mid q \in \Omega_{m,V} \}$ is infinite $\}$.

~~graph $G_{m,V} = (Q,$~~

$q^{I_{m,V}} : \{1, \dots, d\} \to \mathbb{N} \cup \{\infty\}$ where

$q^{I_{m,V}}(i) = q(i)$ iff $i \notin I_{m,V}$

$q^{I_{m,V}}(i) = \infty$ iff $i \in I_{m,V}$

graph $G_{m,V} = (Q, A, E)$, $Q = \{ q^{I_{m,V}} \mid q \in Q_{m,V} \}$

$E = \{ (p^{I_{m,V}}, a, q^{I_{m,V}}) \mid p, q \in Q_{m,V}, q = p + a \}$.

$R_{m,V}$ is the relation defined as $(r, s) \in (\mathbb{N}^d \times \mathbb{N}^d) \cap V$ s.t. $r(i) = s(i) = 0$ for every $i \in \{1, \dots d\} \setminus I_{m,V}$ and such that there exists a cycle in $G_{m,V}$ on the state $m^{I_{m,V}}$ labeled by a word $a_1 \dots a_k$ where $a_j \in A$ s.t. $r + \sum_{j=1}^{k} a_j = s$.

Lemma 8.4

<u>Lemma</u> : $R_{m,V}$ is periodic & Presburger definable.

<u>Proof</u>: Parikh image of ~~Presburger relation~~ regular language is Presburger definable.

QED.

Lemma 8.6

Lemma: $\overline{\mathbb{Q}_{\geq 0} P}$ is finitely generated for every ~~Presburger set P~~ periodic set P.

Proof: $P = \bigcup_{j=1}^{k} b_j + P_j$. $b_j \in \mathbb{Z}^d$. $P_j \subseteq \mathbb{Z}^d$ is finitely generated.

$$C = \sum_{j=1}^{k} (\mathbb{Q}_{\geq 0} b_j + \mathbb{Q}_{\geq 0} P_j).$$

$$P \subseteq C, \quad \mathbb{Q}_{\geq 0} P \subseteq C.$$

C is finitely generated $\Rightarrow$ C is closed.

$$\therefore \overline{\mathbb{Q}_{\geq 0} P} \subseteq C.$$

Let $p \in P_j$. $b_j + np \in P$ $\forall n > 0$

$$\Rightarrow \frac{1}{n} b_j + p \in \mathbb{Q}_{\geq 0} P \quad \forall n > 0.$$

$$\Rightarrow p \in \overline{\mathbb{Q}_{\geq 0} P}$$

$$\Rightarrow P_j \subseteq \overline{\mathbb{Q}_{\geq 0} P} \Rightarrow \mathbb{Q}_{\geq 0} P_j \subseteq \overline{\mathbb{Q}_{\geq 0} P}$$

$$\mathbb{Q}_{\geq 0} b_j \subseteq \mathbb{Q}_{\geq 0} P \subseteq \overline{\mathbb{Q}_{\geq 0} P}.$$

$$\Rightarrow C \subseteq \overline{\mathbb{Q}_{\geq 0} P}.$$

QED.

Lemma 8.5

Lemma: $\overline{\mathbb{Q}_{\geq 0} \xrightarrow{*}_{m,V}} = \overline{\mathbb{Q}_{\geq 0} R_{m,V}}$.

Proof: ~~($\supseteq$)~~ ($\subseteq$): Suppose $r \xrightarrow{*}_{m,V} s$.

$\exists w$: $m + r \xrightarrow{w} m + s$. $m + nr, m + ns \in Q_{m,V}$ $\forall n \in \mathbb{N}_0$

$\therefore$ $r(i) > 0$ or $s(i) > 0$ implies $i \in I_{m,V}$.

$\therefore$ $m^{I_{m,V}} \xrightarrow{w} m^{I_{m,V}}$. $\therefore$ w is the label of an cycle in $G_{m,V}$ on $I_{m,V}$ $\therefore$ $(r, s) \in R_{m,V}$.

(2): An intraproduction is a triple (r, x, s) s.t. $x \in \mathbb{N}^d$, $(r,s) \in (\mathbb{N}^d \times \mathbb{N}^d) \cap V$ and

$$r \xrightarrow{*}_m x \xrightarrow{*}_m s.$$

An intraproduction is total if $x(i) > 0$ for every $i \in I_{m,V}$.

There exists a total intraproduction for (m,V).

[+ b p 1]

NOW, let $(r,s) \in R_{m,V}$. Let ω be the label of the cycle on $m^{I_{m,V}}$ in $G_{m,V}$ s.t. ~~$m+$~~ $r + \sum_{j=1}^{|\omega|} \omega(j) = s$. ~~There exists a~~ ~~large~~ ~~enough~~ p

~~s.t.~~ ~~$m + px + r \xrightarrow{\omega} m + px + s$.~~

~~Since~~ ~~$m + pr' \xrightarrow{*} m + ps'$~~

Let (r', x', s') be a total intraproduction for (m,V). There exists a large enough p s.t.

$m + px + r \xrightarrow{\omega} m + px + s$.

Since $m + pr' \xrightarrow{*} m + ps'$ we get

$m + pr' + nr \xrightarrow{*} m + ps' + ns \quad \forall n \in \mathbb{N}$

Hence $p(r', s') + \mathbb{N}(r, s) \subseteq \xrightarrow{*}_{m,V}$. Thus $(r,s) \in \overline{Q_{>0} \xrightarrow{*}_{m,V}}$.

$$\Rightarrow R_{m,V} \subseteq \overline{Q_{>0} \xrightarrow{*}_{m,V}}$$

$$\Rightarrow \overline{Q_{>0} R_{m,V}} \subseteq \overline{Q_{>0} \xrightarrow{*}_{m,V}}$$

QED

$\boxed{\text{Lemma 8.3}}$

Lemma: There exists a total intraproduction for (m,V).

Proof: Finite sums of intraproductions are intraproductions, so it is sufficient to prove that for every $i \in I_{m,V}$, there exists an intraproduction ~~for (m,V)~~ (r,x,s) for (m,V) such that $x(i)>0$.

$$i \in I_{m,V} \Rightarrow q_1(i) < q_2(i) < \cdots\cdots \in Q_{m,V}.$$

From Dickson's lemma, we get $q \leq q'$ s.t. $q(i) < q'(i)$

$q, q' \in Q_{m,V} \Rightarrow \exists\, (r,s), (r',s') \in (\mathbb{N}^d \times \mathbb{N}^d) \cap V$ s.t.

$$m+r \xrightarrow{*} q \xrightarrow{*} m+s.$$

$$m+r' \xrightarrow{*} q' \xrightarrow{*} m+s'.$$

Let $v = q'-q$.

$\cancel{m+r+r' \xrightarrow{*}}$

$$\underline{m+r'}+r \xrightarrow{*} \overline{q}+(\underline{q'}-q)+\overline{r} \xrightarrow{*} \overline{\underline{m}+s+\underline{r}}+(q'-q) \xrightarrow{*}$$

$$s+\underline{q}+(\overline{q'}-q) \rightarrow \overline{m+s'}+s$$

$$\therefore\ m+r'+r \xrightarrow{*} m+(q'-q)+s+r \xrightarrow{*} m+s'+s.$$

$(r+r', (q'-q)+s+r, s+s')$ is an intraproduction for (m,V) with $(q'-q)(i)>0$. QED.