Topics in Logic and Automata Theory

Logic and Automata over Graphs

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Notations and Symbols

Henceforth we assume the following :-

- σ is the vocabulary σ=(R_1,...,R_m,c_1,...,c_s) where, $\forall i \in \{1, .., m\}, R_i$ is a relation symbol of arity k_i, for some $k_i \in \mathbb{N}$ and $\forall i \in \{1, .., s\}, c_i$ is a unique constant symbol.

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- $\mathcal{G}_A$ and $\mathcal{G}_B$ are the gaiffman graphs (explained in the next slide) for $\mathcal{A}$ and $\mathcal{B}$ respectively.
- Given an element a $\in A$, N($\mathcal{A}$,a)$\upharpoonright$d is the neighbourhood or sphere or subgraph of the Gaiffman graph of $\mathcal{A}$, $\mathcal{G}_A$, with a as center and a radius of d.

Gaiffman Graph

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- A is the domain of the structure $\mathcal{A}$.
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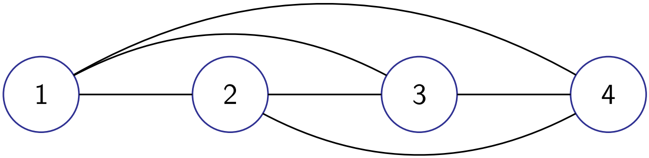

Local Equivalence

Some points and terms to note related to graphs :

- For any d $\in \mathbb{N}$, the number of spheres of radius d is finite.
- Let n $\in \mathbb{N}$, be the number of spheres for a fixed radius d.
- Then we can talk of a type signature of a graph given by ($\#\text{Type}_1$,...,$\#\text{Type}_n$) which is the number of spheres of radius d that are of Type_1,Type_2 ... Type_n.

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Definition (Local d-Equivalence)

Two structures $\mathcal{A}$ and $\mathcal{B}$ are said to be locally d-equivalent for some d $\in \mathbb{N}$, iff both $\mathcal{A}$ and $\mathcal{B}$ have the same type signature of radius d. Let it be denoted by $\mathcal{A} \sim_d \mathcal{B}$.

Logical Equivalence

Definition (Logical r-Equivalence)

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- The above statement holds in both directions namely, $\mathcal{A} \equiv_r \mathcal{B} \Longleftrightarrow$ Duplicator has a winning strategy for the r-round EF game.

Hanf's theorem

Theorem (Hanf's)

Let $d,r \in \mathbb{N}$ *such that* $d \geq 3^{r-1}$.
Then, $\mathcal{A} \sim_d \mathcal{B} \Longrightarrow \mathcal{A} \equiv_r \mathcal{B}$.

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Proof sketch :-

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Proof sketch :-

- The duplicator's strategy in Round 1

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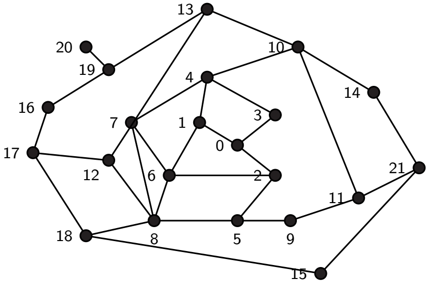
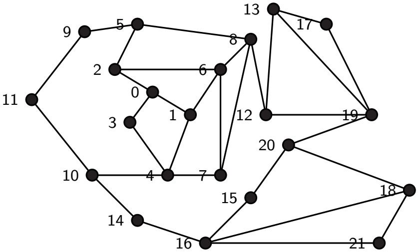
- Variations of hanf's theorem

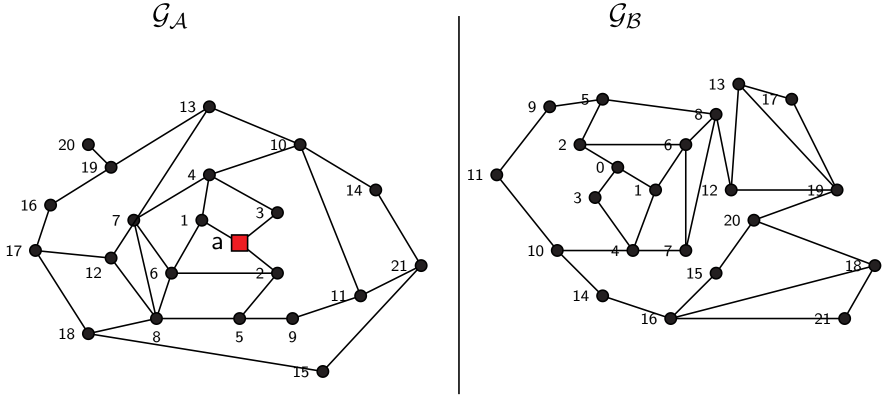
The EF-Game Graphs

$\mathcal{G}_{\mathcal{A}}$



$\mathcal{G}_{\mathcal{B}}$





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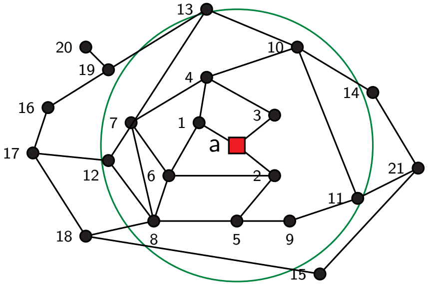
Round 1

$\mathcal{G}_{\mathcal{A}}$



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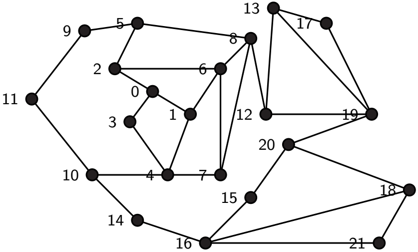

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- The d-neighbourhood of a in $\mathcal{A}$, denoted N($\mathcal{A}$,a)$\restriction$d is of one of the types $\{\text{Type}_1, \ldots \text{Type}_n\}$

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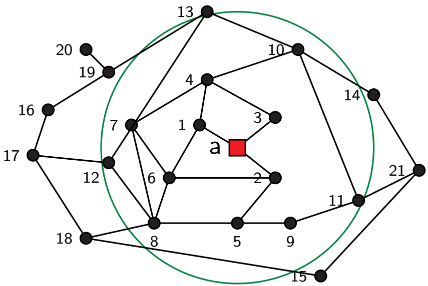

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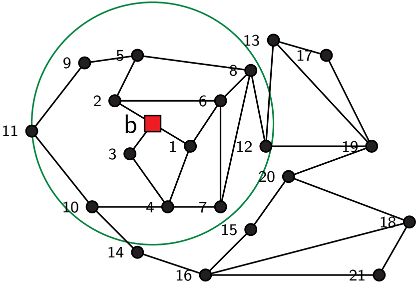

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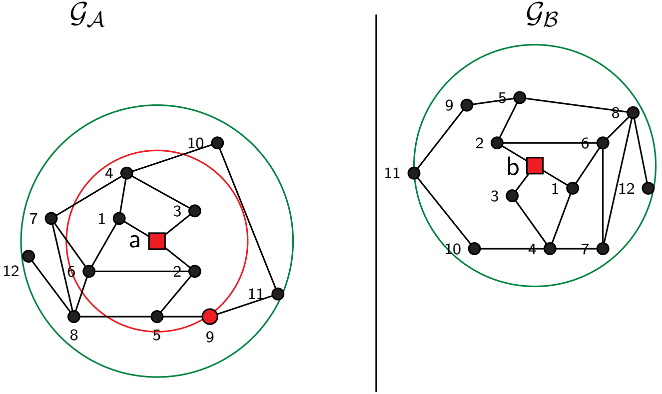
- The d-neighbourhood of a in $\mathcal{A}$, denoted N($\mathcal{A}$,a)$\restriction$d is of one of the types $\{\text{Type}_1, \ldots\ \text{Type}_n\}$
- The Duplicator picks an element from the other structure (here, $b \in \mathcal{B}$) such that N($\mathcal{A}$,a)$\restriction$d $\cong$ N($\mathcal{B}$,b)$\restriction$d

Round i, Case 1



- In Round i, the first case is when the Spoiler picks a vertex that is within $2*3^{i-2}$ of any previously selected point (maybe more than one).

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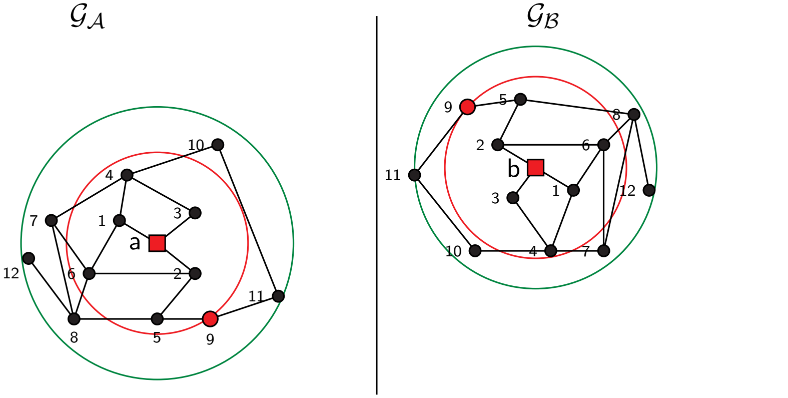

- In Round i, the first case is when the Spoiler picks a vertex that is within $2*3^{i-2}$ of any previously selected point (maybe more than one).
- Then the Duplicator will use the isomorphism of the d-radius sphere around any one of the centres to obtain a similar vertex on the other graph.

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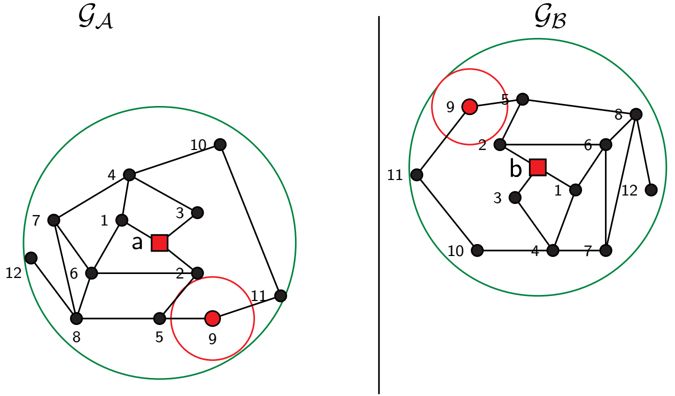

- The reason why it works is because, as shown in the figure, in subsequent i-1 rounds the Spoiler will not be able to get out of the isomorphism of the d-sphere around the previously selected points.

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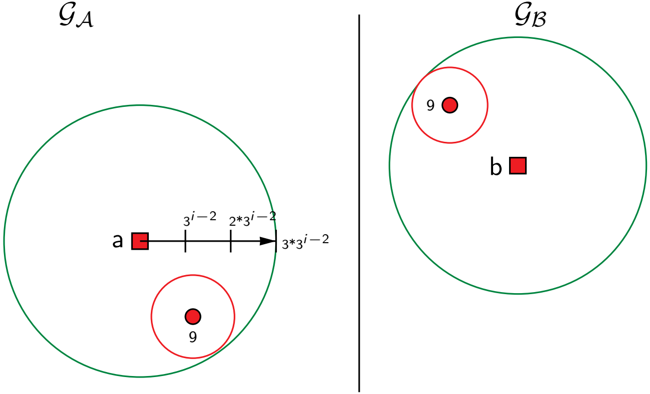
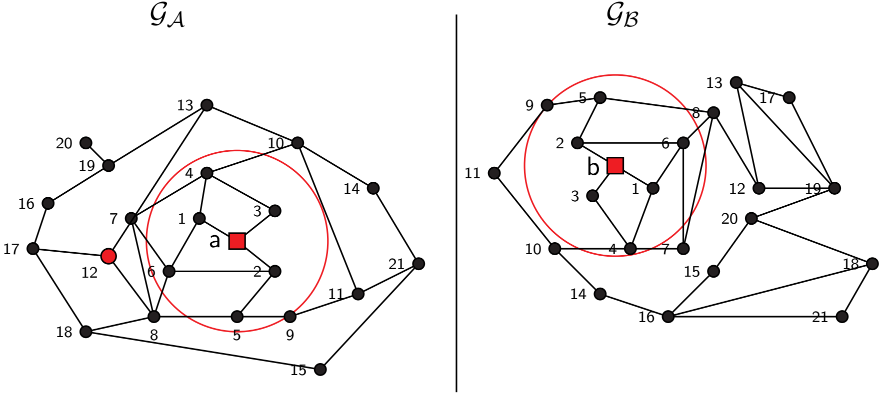


- The reason why it works is because, as shown in the figure, in subsequent i-1 rounds the Spoiler will not be able to get out of the isomorphism of the d-sphere around the respective previously selected points.
- This is due to the fact that $\forall$ i $\in \mathbb{N}$, $3^{i-2} \geq$ (i-1).

Round i, Case 2



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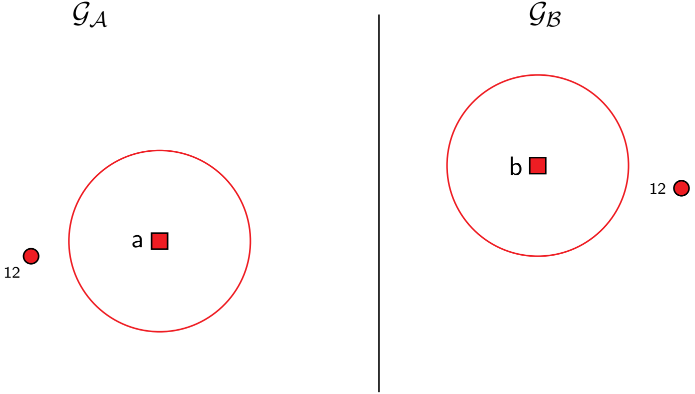

- Now, the other case is if the Spoiler picks a vertex that is outside $2*3^{i-2}$ of **all** previously selected points.
- Then, as in Round 1, the Duplicator will be able to pick a vertex $b \in \mathcal{B}$ such that $N(\mathcal{A},a)\restriction d \cong N(\mathcal{B},b)\restriction d$. Also, this particular point b is not in the range of $2*3^{i-2}$ distance of any other previously selected point in $\mathcal{G}_{\mathcal{B}}$.

Variations of Hanf's theorem

Definition

Given d,t $\in \mathbb{N}$, we can define the concept of type signatures of radius d with threshold t such that the values (#Type_1,...,#Type_n) are counted only upto a threshold t and anything $\geq$ t is considered ∞. Two structures $\mathcal{A}$ and $\mathcal{B}$, are said to be d-equivalent with threshold t if their type signatures with radius d are equal. It is denoted $\mathcal{A} \sim_{d,t} \mathcal{B}$.

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Theorem (Hanf's theorem for EMSO)

- *Let ϕ be an EMSO formula with n second-order quantifiers given by, $\phi = \exists X_1 \ldots \exists X_n \psi(X_1, \ldots, X_n)$, where ψ is a pure first order sentence.*

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- *Let ϕ be an EMSO formula with n second-order quantifiers given by, $\phi = \exists X_1 \ldots \exists X_n \psi(X_1, \ldots, X_n)$, where ψ is a pure first order sentence.*
- *If we consider the extended models of $\mathcal{A}$, $\mathcal{A}' = (A \times 2^{\{0,1,\ldots k\}}, R_1^A, \ldots, R_m^A, c_1^A, \ldots, c_s^A)$, then we can reuse Hanf's theorem as only the ψ part remains to be interpreted over these modified structures.*

MSO and automata over pictures

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Definition (Pictures, Picture Languages)

- A picture, p, over an alphabet Σ is basically a function of the form
  $$p : \{1,2,\ldots, n\} \times \{1,2,\ldots,m\} \rightarrow \Sigma, \text{ for any } n,m \in \mathbb{N}$$
- The set of all pictures (over Σ) is the set of all possible functions p for every n,m $\in \mathbb{N}$. It is denoted by Σ^{**}
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p :

a	b
b	a

q :

a	a	b
a	b	a

r :

a	a	a	a	a	a	a
b	b	b	b	b	b	b
a	a	a	a	a	a	a
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- A picture language is a subset of Σ^{**}
- If p is a picture of size (m,n), then $\widehat{p}$ is the picture p surrounded by a special boundary symbol $\# \notin \Sigma$

$\hat{p}$:

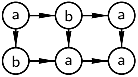
#	#	#	#
#	a	b	#
#	b	a	#
#	#	#	#

$\hat{r}$:

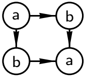
#	#	#	#	#	#	#	#	#
#	a	a	a	a	a	a	a	#
#	b	b	b	b	b	b	b	#
#	a	a	a	a	a	a	a	#
#	b	b	b	b	b	b	b	#
#	a	a	a	a	a	a	a	#
#	#	#	#	#	#	#	#	#

Row Concatenation of 2 pictures

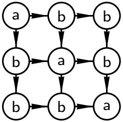
p :



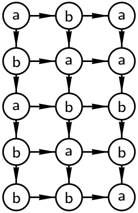
q :



r :



$p \ominus r$:

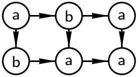


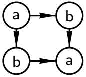
$p \ominus q$: Undefined

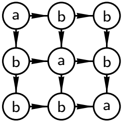
$q \ominus r$: Undefined

(As the number of columns are incompatible)

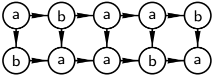
Column Concatenation of 2 pictures

p :


q :


r :


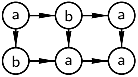
p ⊘ r : Undefined
q ⊘ r : Undefined

p ⊘ q:


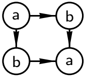
(As the number of rows are incompatible)

Row and Column Concatenation of 2 pictures

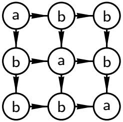
p :



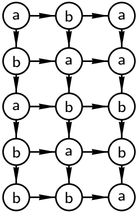
q :



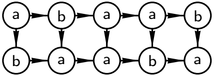
r :



$p \ominus r$:



$p \oslash q$:



Operations on Picture Languages

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- Given a picture p of size (m,n), if $h \leq m$, $k \leq n$, we denote by $T_{h,k}(p)$ the set of all subpictures (contiguous rectangular subblocks) of p of size (h,k).

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#	#	#	#	#	#
#	1	0	0	0	#
#	0	1	0	0	#
#	0	0	1	0	#
#	0	0	0	1	#
#	#	#	#	#	#

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A picture language L $\subseteq \Sigma^{**}$ is recognizable if there exists a local language L' over an alphabet Γ and a mapping

$\pi : \Gamma \rightarrow \Sigma$ such that L$=\pi(L')$

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- Under the above considerations, the tiling System is denoted by the triple (Σ,Q, Δ).

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  - The claim is that L $\notin$ REC while $\overline{L} \in$ REC.

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A few notations and terminologies :

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- If ϕ is a sentence we write $\underline{p} \models \phi$.

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Definition (EMSO-definable)

Finally, A picture language L is existential monadic second-order definable ($L \in EMSO$), if there is a sentence of the form

$$\phi = \exists X_1 \ \ldots \ \exists X_n \psi(X_1, \ldots, X_n) \text{ where } \psi \text{ contains only first-order}$$

quantifiers such that $L = L(\phi)$.

REC $\iff$ EMSO

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- While $\chi_m,\chi_t,\chi_b,\chi_l,\chi_r,\chi_{tl},\chi_{tr},\chi_{bl},\chi_{br}$ refer to the formulae describing (2,2) local neighbourhoods.

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- If it holds for some d and t then we say L is **locally threshold testable**.
- Finally, if L is a union of $\simeq_{d,t}$-classes for some t, L is called **locally strictly threshold d-testable**.

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A picture language is first-order definable iff it is locally threshold testable.

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- The proof for the direction $\Longleftarrow$ is by an adaptation of Hanf's theorem to pictures. We can use the bound as $d=2*3^n+1$ and $t=n*3^{2n}$ for n-equivalence.

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First-order definable $\Longrightarrow$ Locally threshold testable

Proof Sketch :-

Theorem

Each locally threshold d-testable language L can be decomposed into $L_0 \cup L_1 \cup \ldots \cup L_{d-2}$ *where* $L_i \subseteq \Sigma_i^{**}$ *(0≤i≤d-2) is locally strictly threshold (i+2)-testable.*

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Definition

Let $d \geq 2$ be a positive integer. A picture language $L \subseteq \Sigma_{d-2}^{**}$ is d-local if there exists a set $\Delta_{(d)}$ of pictures of size (d,d) (or "d-tiles") over $\Sigma \cup \{\#\}$, such that $L = \{p \in \Sigma^{**} \mid T_{d,d}(\widehat{p}) \subseteq \Delta^{(d)}\}$

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Let $d \geq 3$ be a positive integer. A locally strictly threshold d-testable picture language $L \subseteq \Sigma_{d-2}^{**}$ is the projection of a d-local language.

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Alternation hierarchy of MSO over grids and graphs

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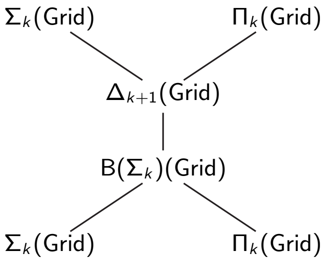
$\forall\ k \geq 1,\ B(\Sigma_k)(Grids) \subsetneq \Delta_{k+1}(Grids)$

The inclusion results are as shown the diagram in the right with undirected edges indicating strict inclusion.



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The basis of the theorem is definability results for sets of grids.

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Let $f_1(m) = 2^m$, $f_{k+1}(m) = f_k(m)2^{f_k(m)}$ for m,k $\geq$ 1.
$\forall k \geq 1$,the function f_k is definable in Σ_k and Π_k over τ_{Grid}

Complexity of a $B(\Sigma_k)$-definable function

Proof Sketch:

- For a picture language L over alphabet Γ and an integer m$\geq$ 1, we denote by L(m) the word language L restricted to $\Gamma^{m,1}$.

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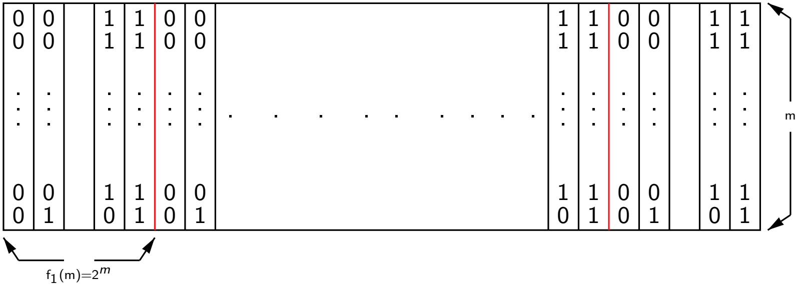
- Let ϕ be a $B(\Sigma_k)$-sentence. There is a constant $c \geq 1$ such that for every $m \geq 1$ the set $Mod_0(\phi)(m)$ is $s_k(cm)$-periodic. Thus from all the above statements we get that every $B(\Sigma_k)$-definable function is at most k-fold exponential.

$f_k(m) \in \Delta_k(\text{Grid})$



The Complete column-numbering of a grid

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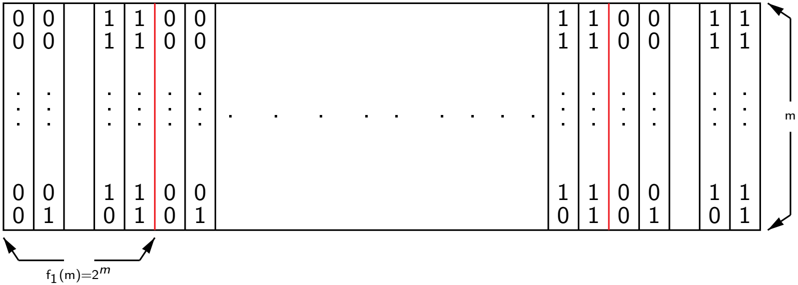

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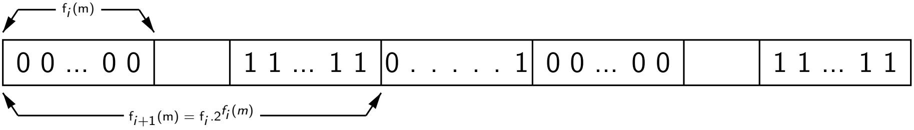

Complete f_i-numbering along the top row of the grid

MSCs, MPA and EMSO_{MSC}

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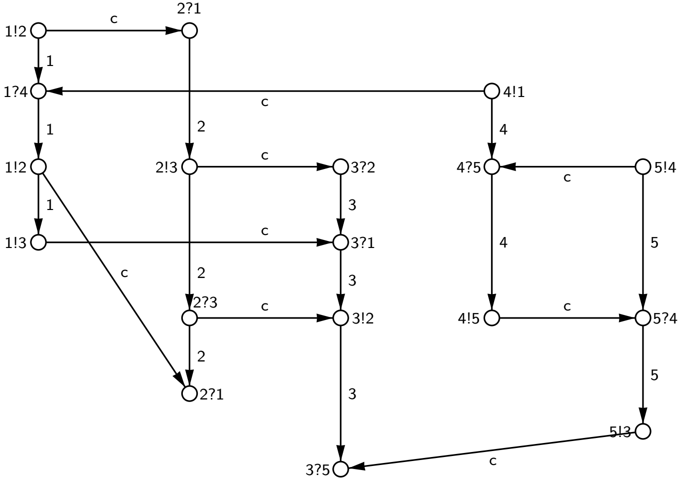
Message Sequence Charts :-



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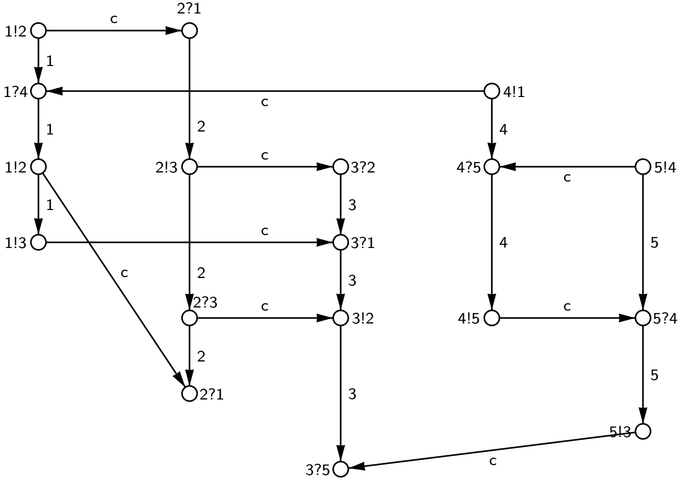

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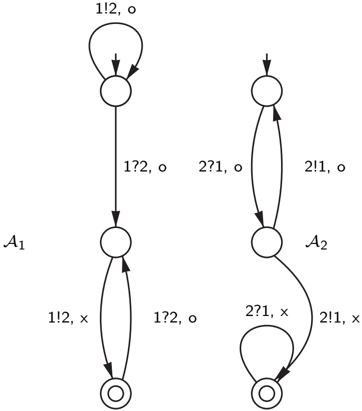

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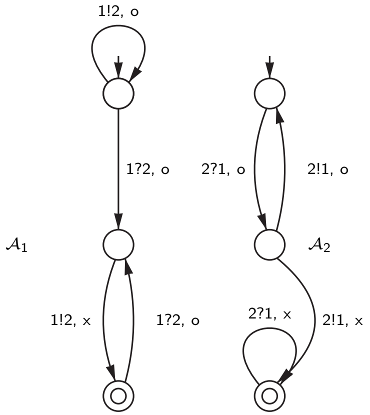

MPA (over P) is a structure $(A) = (((A)_p)_{p \in P}, \mathcal{D}, s^{-in}, F)$

Definitions

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An MSC (over P) is a graph $M = (E,\{\Delta_p\}_{p \in P},\Delta_c,\lambda) \in \mathbb{DG}(Act,P_c)$ such that

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MPA (over P) is a structure $(A) = (((A)_p)_{p\in P},\mathcal{D},\text{s}^{-in},\text{F})$ such that

- $\mathcal{D}$ is a set of synchronization messages.
- for each p∈P ,$\mathcal{A}$ is a pair (S_p, δ_p) where
  - S_p is a set of local states
  - $\delta_p \subseteq S_p \times \text{Act}_p \times \mathcal{D} \times S_p$
- $\text{s}^{-in} \in \Pi_{p\in P}\text{S}_p$ is the global initial state.
- $\text{F} \subseteq \Pi_{p\in P}\text{S}_p$ is the set of global final states.

Logical Definability

Definition (MSO(Σ,C))

MSO(Σ,C) over the class $\mathbb{DG}$ are built up from the atomic formulas $\lambda(x) = a$ (for $a \in \Sigma$), $x\Delta_c y$ (for $c \in C$), $x \in X$ and $x = y$.

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Definition

A graph acceptor over (Σ,C) is a structure $\mathcal{B} = (\mathcal{Q}, \mathcal{R}, \widehat{S}$,Occ) where

- $\mathcal{Q}$ is its nonempty finite set of states
- $\mathcal{B} \in \mathbb{N}$ is the radius
- $\widehat{S}$ is a finite set of R-spheres over ($\Sigma \times \mathcal{Q}$,C) and
- Occ is a boolean combinations of conditions of the form "sphere H $\in \widehat{S}$ occurs at least n times" where n $\in \mathbb{N}$.

The Theorems

Theorem

$MPA \equiv EMSO_{MSC}$

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Theorem

The monadic quantifier alternation hierarchy over MSC is infinite.

Summary

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- Hanf's theorem
- Picture languages
  - Local picture Languages
  - Recognizable picture languages
  - REC $\iff$ EMSO
- Monadic quantifier Alternation hierarchy over Grids and graphs
- MSC $\iff$ EMSO_{MSC}