Existence of fibred product of Schemes

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Contents

Pre-sheaf

First we shall define presheaf (for topological spaces).

Definition 1.1 (Pre-sheaf)

Consider X be a topological space then a presheaf $\mathcal{F}$ of abelian groups on X consists of the data:

1. *For every open subset $U \subseteq X$ an abelian group $\mathcal{F}(U)$(also denoted by $\Gamma(U, F)$ and called section)*
2. *For every inclusions $V \subseteq U$ of subsets of X, a morphism of abelian groups $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$. Note sometime we denote this map by $s|_V = \rho_{UV}(s)$.*

subject to the conditions:

1. *$\mathcal{F}(\phi) = 0$, ϕ being the empty set.*
2. *ρ_{UU} is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$*
3. *If $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ (contravariance property)*

Sheaves

Definition 1.2 (Sheaf)

A presheaf $\mathcal{F}$ on a topological space X is a sheaf $\iff$ it satisfies:

1. *For every open set U if $\{V_i\}$ is a open covering of U and if $s \in \mathcal{F}(U)$ is an element such that $s|_{V_i} = 0$ for all i then $s = 0$.*
2. *For every open set U if $\{V_i\}$ is an open covering of U and if we have elements of $s_i \in \mathcal{F}(V_i)$ for each i with the property that for each i, j $s_{i|V_i \cap V_j} = s_{j|V_i \cap V_j}$ then there is an element $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for each i (glueing property).*
   Note that s is unique from the first property.

Definition 1.3 (Stalk)

$\mathcal{F}$ is a presheaf on X and if P is a point of X we define the stalk $\mathcal{F}_P$ of $\mathcal{F}$ at the point P to be the direct limit of the groups $\mathcal{F}(U)$ for all open sets U containing P by the restricting maps ρ.

Spectrum

Definition 1.4 (Sheaf of rings)

For each prime ideal $p \in A$ denote A_p be the localization of A at p. For an open set $U \subseteq Spec\ A$, we define $\mathcal{O}(U)$ to be the set of functions $s : U \to \coprod_{p \in U} A_p$ such that $s(p) \in A_p$ for each p and such that s is locally a quotient of elements of A. Note that sums and products of such functions are again such functions and the element 1 that gives 1 in each A_p is an identity thus making $\mathcal{O}(U)$ a commutative ring with identity. If $V \subseteq U$ are two open sets, the natural restriction map $\mathcal{O}(U) \to \mathcal{O}(V)$ is a homomorphism of rings making $\mathcal{O}$ a presheaf and it is easy to check: this in fact it is sheaf because of the local nature of the definition.

Definition 1.5 (Spectrum)

Consider A to be a ring. The spectrum of the ring A is the pair consisting of the topological space $Spec\ A$ together with the sheaf of rings $\mathcal{O}$.

Schemes I

Schemes enlarges the notion of algebraic variety in ways such as taking account of multiplicities and allowing varieties defined over any commutative ring.

Definition 1.6 (Locally ringed space)

A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X.
The ringed space $(X, \mathcal{O}_X)$ is a locally ringed space if for each point $P \in X$, the stalk $\mathcal{O}_{X,P}$ is a local ring (unique maximal ideal).

Schemes II

Definition 1.7 (Morphisms)

A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^{\#})$ of continuous map $f : X \to Y$ and a map $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ (where f_* is the direct image) of sheaves of on Y.
If A and B are local rings with maximal ideals m_A and m_B respectively, then a homomorphism $\phi : A \to B$ is called a local homomorphism if $\phi^{-1}(m_B) = m_A$.
A morphism of locally ringed space is a morphism $(f, f^{\#})$ of the ringed spaces such that for each point $P \in X$ the induced map of local rings $f_P^{\#} : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is a local homomorphism of local rings.

Schemes III

Definition 1.8 (Schemes)

An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ *which is isomorphic (as a locally ringed space) to the spectrum of some ring. A scheme is a locally ringed space* $(X, \mathcal{O}_X)$ *(*$\mathcal{O}_X$ *is called the structure sheaf) in which every point has an open neighbourhood* U *such that the topological space* U*, together with the restricted sheaf* $\mathcal{O}_{X|U}$*, is an affine scheme.*
A morphism of schemes is a morphism as locally ringed spaces. An isomorphism is a morphism with a two-sided inverse.

Definition 1.9 (Open subscheme)

An open subscheme of a scheme X *is a scheme* U*, whose topological space is an open set subset of* X*, and whose structure sheaf* $\mathcal{O}_U$ *is isomorphic to the restriction* $\mathcal{O}_{X|U}$ *of the structure sheaf of* X*.*

Properties of Schemes I

Theorem 2.1

Consider a ring A and $(X, \mathcal{O}_X)$ be a scheme. Now given a morphism $f : X \to Spec\ A$ we have an associated map on sheaves $f^{\#} : \mathcal{O}_{Spec\ A} \to f_*\mathcal{O}_X$(direct image). Taking the global sections we obtain a homomorphism $A \to \Gamma(X, \mathcal{O}_X)$. Then the natural map

$$\alpha : Hom_{Sch}(X, Spec\ A) \to Hom_{Ring}(A, \Gamma(X, \mathcal{O}_X))$$

is bijective.

Properties of Schemes II

Theorem 2.2 (Glueing lemma)

Consider $\{X_i\}$ be a collection of schemes (possibly infinite). For each $i \neq j$ suppose that given an open subset $U_{ij} \subseteq X_i$ and it has the induced scheme structure $(X, \mathcal{O}_{X_i|U_{ij}})$. Also assume that given for each $i \neq j$ an isomorphism of schemes $\phi_{ij} : U_{ij} \to U_{ji}$ such that:

1. $\phi_{ij} = \phi_{ji}^{-1}$
2. $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$
3. $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ *on* $U_{ij} \cap U_{ik}$ *for all* i, j, k

Then there is a scheme X and morphism $\psi_i : X_i \to X$ for each i such that

1. ψ_i *is an isomorphism of* X_i *onto an open subscheme of* X
2. *the* $\psi_i(X_i)$ *cover* X
3. $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$
4. $\psi_i = \psi_i \circ \phi_{ij}$ *on* U_{ij}

Properties of Schemes III

In this case we say that X is glueing the schemes X_i along ϕ_{ij}. Refer to the commutative diagram figure 1.

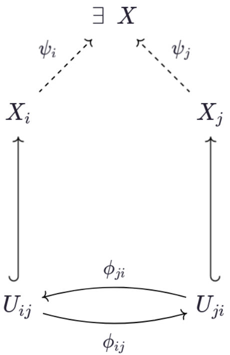

Figure: Glueing lemma

Fibred products

Definition 3.1 (Fibred product)

S is a scheme and X, Y are schemes over S (that is schemes with morphisms to S) then we define the fibred product of X and Y over S denoted $X \times_S Y$ to be a scheme, together with morphisms $p_1 : X \times_s Y \to X$ and $p_2 : X \times_s Y \to Y$ which make a commutative diagram with the given morphisms $X \to S$ and $Y \to S$ such that given any scheme Z over S and given morphisms $f : Z \to X$ and $g : Z \to Y$ which make a commutative diagram with morphisms $X \to S$ and $Y \to S$ then there exists a unique morphism $\theta : Z \to X \times_S Y$ such that $f = p_1 \circ \theta$ and $g = p_2 \circ \theta$. The morphism p_1 and p_2 are called the projection morphisms of the fibred product onto its factors.

Commutative diagram for fibred products

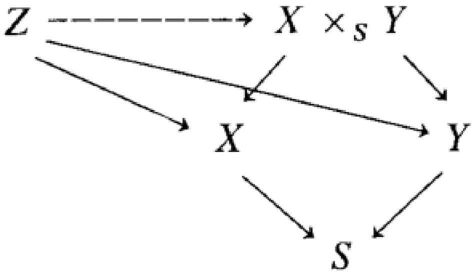

Figure: Fibred morphism

Main theorem

Theorem 4.1 (Existence theorem for fibred product)

For any two schemes X and Y over a scheme S the fibred product $X \times_S Y$ exists and is unique up to unique isomorphism (universal property). [Har77]

Proof I

The idea is to construct the products of affine schemes and then glue them together, we shall do that in the following steps:

Proof II

Step 1 <u>Affine case</u>

Consider the affine case first that is with $X = Spec\ A$, $Y = Spec\ B$ and $S = Spec\ R$. Then A and B are R-algebras. Now we claim that $Spec(A \otimes_R B) = X \times_S Y = Spec\ X \times_{Spec\ R} Spec\ Y$. Now for any scheme Z to give a morphism of Z to $Spec(A \otimes_R B)$ is the same as to give a homomorphism of the ring $A \otimes_R B$ into the ring $\Gamma(Z, \mathcal{O}_Z)$ (theorem 2.1). But to give a homomorphism of the ring $A \otimes_R B$ into any ring is the same as to give homomorphisms of A and B into that ring, inducing the same homomorphism on R. Applying (theorem 2.1) again we see that to give a morphism of Z into $Spec(A \otimes_R B)$ is the same as giving morphisms of Z into X and into Y which give rise to the same morphism of Z into S, thereby we get $Spec(A \otimes_R B)$ as the desired product using bijectivity of the natural map α in theorem 2.1.

Proof III

Step 2 Uniqueness

Now it follows immediately from the universal property of the product that it is unique upto a unique isomorphism whenever it exists. This will be required in subsequent steps.

Proof IV

Step 3 Glueing morphisms

If X and Y are schemes, then to give a morphism f from $X \to Y$, it is equivalent to give an open cover $\{U_i\}$ of X, together with morphisms $f_i : U_i \to Y$, where U_i has the induced open subscheme structure such that the restriction of f_i to $U_i \cap U_j$ are the same, $\forall i, j$. So we can prove this as follows: First consider the easier direction that given $f : X \to Y$ then for every open covering $\{U_i\}$ of X we can find morphism $f_i : U_i \to Y$ (U_i having the induced open subscheme structure) that agrees on the intersection as scheme morphisms. To show this we consider *restrictions*. Explicitly scheme morphisms are $f = (f, f^{\#})$ so $f_i := f|_{U_i}$ and taking the map $f_i^{\#}$ as $f_i^{\#} : Y \to f_* \mathcal{O}_{X|U_i}$ makes it agree on the intersection. Indeed the set theoretic map f_i agrees on the intersection now the map $f_i^{\#}$ that is to say that for every open set $A \subseteq U_i \cap U_j$ we have $f_i^{\#}|_A = f_j^{\#}|_A$ because $f_i^{\#}|_A : Y \to f_* \mathcal{O}_{(X|U_i)|A} = f_* \mathcal{O}_{X|A}$ thus establishing one direction.

Proof V

Now we prove that given the morphisms $f_i : U_i \to Y$ which agrees on the intersection we get a morphism $f : X \to Y$. Again the set theoretic map from topological space $X \to Y$ clearly exists because we define $\forall x \in X$ since $\exists i | x \in U_i \quad f(x) := f_i(x)$ and this is well defined and continuous as they agree on the intersection. We need to show that $f^{\#}$ exists. We know that $f_i^{\#} : \mathcal{O}_Y \to f_{i*}U_i$ exists where $f_{i*}U_i(V) = \mathcal{O}_{U_i}(f_i^{-1}(V))$ and V is an open subset of Y . Now one can define that $f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$ as f_is agrees on the intersection and thus establishing the other side.

Proof VI

Step 4 Local product

Now if X, Y are schemes over a scheme S, if $U \subseteq X$ is an open subset and if the product $X \times_S Y$ exists then we claim that $p_1^{-1}(U) \subseteq X \times_S Y$ is a product for U and Y over S. Given a scheme Z, and morphisms $f : Z \to U$ and $g : Z \to Y$ determine a map of $Z \to X$ by composing with the inclusion $U \subseteq X$. Hence there is a map $\theta : Z \to X \times_S Y$ compatible (i.e. makes the diagram commute) with f, g, and the projections. But since $f(Z) \subseteq U$, we have $\theta(Z) \subseteq p_1^{-1}(U)$. So θ can be regarded as a morphism $Z \to p_1^{-1}(U)$. It is clearly unique (using the universal property), so $p_1^{-1}(U)$ is a product $U \times_S Y$. Refer to the commutative diagram in figure 3.

Proof VII

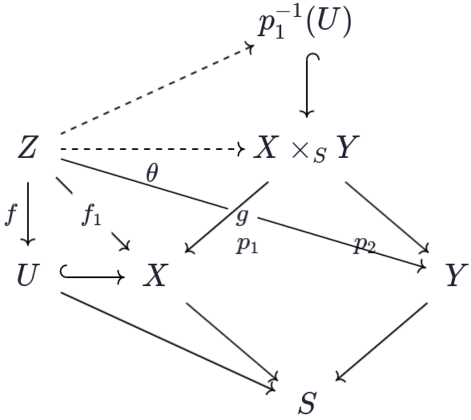

Figure: Step 4

Proof VIII

Step 5 Local (cover) to global

Suppose given X, Y schemes over S, suppose $\{X_i\}$ is an open covering of X and suppose that for each i, $X_i \times_S Y$ exists. Then we claim that $X \times_S Y$ exists. Indeed, for each i, j assume $U_{ij} \subseteq X_i \times_S Y$ be $p_1^{-1}(X_{ij})$, where $X_{ij} = X_i \cap X_j$, then using Step 4, U_{ij} is a product for X_{ij} and Y over S. Hence by the uniqueness of products there are unique isomorphisms $\phi_{ij} : U_{ij} \to U_{ji}$ for each i, j compatible with all the projections in the commutative diagram 4 that is $\phi_{ij} = \phi_{ji}^{-1}$ and $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all i, j, k. Moreover, these isomorphisms are compatible with each other for each i, j, k in the sense of glueing lemma (theorem 2.2) that is $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_{ij} \cap U_{ik}$ for all i, j, k from the commutative diagram 4. Thus we are in a position to glue the schemes using the isomorphism ϕ_{ij} and obtain the scheme $X \times_S Y$ (using glueing lemma theorem 2.2).

Proof IX

That is that is there is a morphism $\psi : X_i \times_S Y \to X \times_S Y$ for each i satisfying

1. ψ_i is an isomorphism of $X_i \times_S Y$ onto an open subscheme of X
2. $\psi_i(X_i)$s cover X
3. $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and
4. $\psi_i = \psi_i \circ \phi_{ij}$ on U_{ij}.

Refer to the commutative diagram figure 4.

Proof X

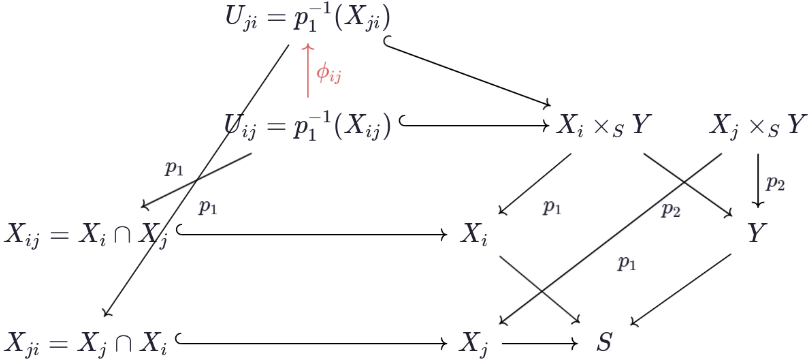

Figure: Step 5: Glueing

Proof XI

Now we claim that this glued scheme $X \times_S Y$ is indeed the fibred product (that is check the morphism properties) for X and Y over S. The projection morphisms p_1 and p_2 are defined by glueing the projections from the pieces $X_i \times_S Y$ (Step 3). Given a scheme Z and morphisms $f : Z \to X,\ g : Z \to Y$ consider $Z_i = f^{-1}(X_i)$. Then we get maps $\theta_i : Z_i \to X_i \times_S Y$, hence by composition with the inclusions $X_i \times_S Y \subseteq X \times_S Y$ we get maps $\theta_i : Z_i \to X_i \times_S Y$. It is easy to see that these maps agree on $Z_i \cap Z_j$, so we can glue the morphisms (Step 3) to obtain a morphism $\theta : Z \to X \times_S Y$, compatible with the projections and f and g. The uniqueness of θ is checked locally, that is since locally it is unique by the universal property it is globally unique. Refer to the commutative diagram figure 5. This proves the morphism properties and hence proving that $X \times_S Y$ is indeed the fibred product.

Proof XII

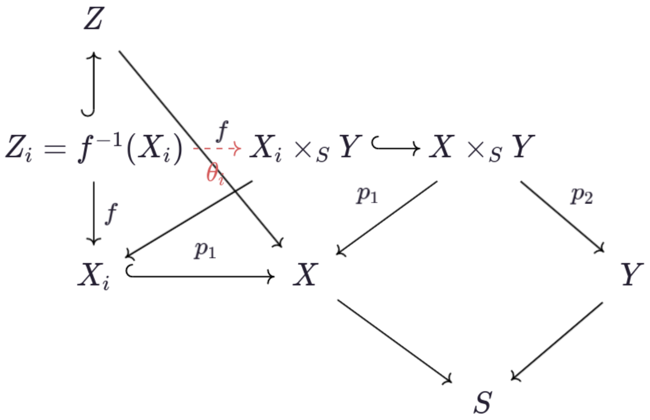

Figure: Step 5: Product

Proof XIII

Step 6 Some logical manipulations

We know from Step 1 that if X, Y, S are all affine, then $X \times_S Y$ exists. Thus using Step 5 we conclude that for any X but Y, S affine, the product exists. Thus using Step 5 again with X and Y interchanged, we find that the product exists for any X and any Y over an affine S.

Proof XIV

Step 7 <u>Global existence</u>

Given arbitrary X, Y, S consider the morphisms $q : X \to S$ and $r : Y \to S$ be the given morphisms. Assume S_i be an open affine cover of S. Consider $X_i := q^{-1}(S_i)$ and $Y_i = r^{-1}(S_i)$ then by using Step 6, $X_i \times_{S_i} Y_i$ exists and note that $X_i \times_{S_i} Y_i = X_i \times_S Y$. This is because given morphisms $f : Z \to X_i$ and $g : Z \to Y$ over S the image of g shall be inside Y_i and thus $\forall i \;\; X_i \times_S Y$ exists. Now one more application of Step 5 gives us $X \times_S Y$ hence completing the proof. Refer to the commutative diagram figure 6.

Proof XV

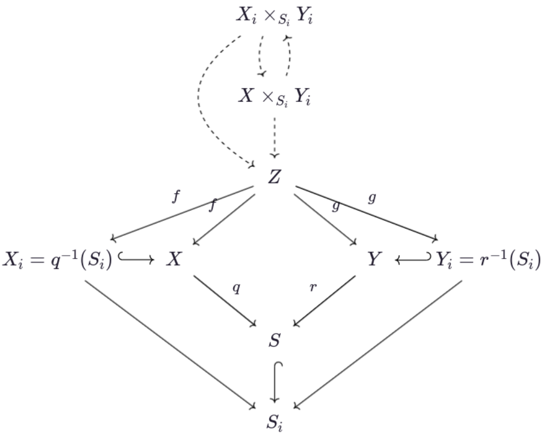

Figure: Step 7: General base S

Bibliography

[Har77] Robin Hartshorne. "Graduate texts in mathematics". In: *Algebraic Geometry* 52 (1977).

Thank you!