Fluid Dynamics, Autumn 2024, CMI
Assignment 3
Due by the beginning of the class on Monday, Sep 9, 2024
Stream function, vector identities

1. $\langle \mathbf{4{+}1} \rangle$ Suppose a flow $\boldsymbol{v}(x,y,t) = (u,v,0)$ on the x-y plane is incompressible. (a) Express the velocity and vorticity ($\boldsymbol{w} = \nabla \times \boldsymbol{v}$) fields in terms of a stream function ψ. (b) Suppose the flow is irrotational as well. What condition does this impose on the stream function?

2. $\langle \mathbf{4} \rangle$ Gauss' divergence theorem relates the volume integral (over a 3d region Ω) of the divergence of a vector field $\boldsymbol{q}$ to the flux of the vector field across its bounding surface $\partial\Omega$. The outward pointing normal on $\partial\Omega$ is denoted $\hat{\boldsymbol{n}}$. Use Gauss' divergence theorem to obtain the following Corollary of the divergence theorem for any scalar function p:

$$\int_{\Omega} \nabla p \, d^3\boldsymbol{r} = \int_{\partial\Omega} p \, \hat{\boldsymbol{n}} \, dA. \tag{1}$$

3. $\langle \mathbf{6} \rangle$ Establish the **vector identity**

$$(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \nabla \left(\frac{1}{2} \boldsymbol{v} \cdot \boldsymbol{v} \right) + (\nabla \times \boldsymbol{v}) \times \boldsymbol{v}, \tag{2}$$

by finding the i^{th} component of the left and right sides. Hint: Use the identity $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.