Notes on Fluid Dynamics, CMI, Autumn 2024

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Course website http://www.cmi.ac.in/~govind/teaching/fluid-dyn-o24

Contents

1 Introduction to mechanics of deformable media

Continuum mechanics begins by dealing with the nonrelativistic classical dynamics of continuous deformable media. Examples are oscillations of stretched strings, heat conduction in rods, elastic motion of solids (rods/beams), motion of fluids[1] (air, water) and plasmas (ionized gases), in roughly increasing order of complexity. All of these systems involve a very large number of molecules (or degrees of freedom) and we will treat them as continuous mass/charge distributions with an infinite number of degrees of freedom. Thus, unlike particle mechanics, continuum mechanics deals with fields. Examples of fields include the height of a stretched string, temperature, elastic displacement, mass density, velocity, pressure, internal energy, specific entropy, charge density, current density, electric and magnetic fields. While a classical point particle is *somewhere* at any given time, a classical field is *everywhere* at any given instant! Thus, continuum mechanics is a collection of (primarily nonrelativistic, classical) field theories. Electromagnetism and gravitation are other examples of field theories, though they often involve relativistic and/or quantum effects.

Due to the larger number of degrees of freedom, the dynamics of deformable bodies is generally more complicated than that of point particles or rigid bodies. In fact, we can imagine a rigid body becoming deformable by relaxing the constraints that fix the distances between its constituents.

[1]Collectively, sand grains sometimes flow like a fluid, though individual grains display properties normally associated with a solid.

There are two principal formalisms for treating mechanics of continuous media, the so-called Lagrangian and Eulerian descriptions. The former is closer to our treatment of systems of particles: we follow the motion of each molecule or fluid element (to be defined in Sect. 3) or bit of string. For example, if a fluid element occupied the location $\boldsymbol{a}$ at $t = 0$, then we seek the trajectory $\boldsymbol{r}(\boldsymbol{a}, t)$ of this fluid element, which should be determined by Lagrange's equations (ironically, this treatment was originally attempted by Euler). The Lagrangian description is particularly useful if we have some way of keeping track of which material element is where. This is usually not possible in a flowing liquid or gas, but is possible in a vibrating string since the bits of string are ordered and may be labeled by their location along the string or by their horizontal coordinate x for small vertical vibrations of a string that does not 'bend over'. For an elastic solid, the corresponding variable is the local displacement field $\boldsymbol{s}(\boldsymbol{r}, t)$ or $\boldsymbol{\xi}(\boldsymbol{r}, t)$ which represents the departure from the equilibrium location of the element that was originally at $\boldsymbol{r}$. In a fluid like air or water, it is difficult to follow the motion of individual fluid elements due to the tendency to mix.

So Euler developed the so-called Eulerian description, which attempts to understand the dynamics of quantities (Eulerian variables) such as density $\rho(\boldsymbol{r}, t)$, pressure $p(\boldsymbol{r}, t)$, velocity $\boldsymbol{v}(\boldsymbol{r}, t)$ and temperature $T(\boldsymbol{r}, t)$ in a fluid at a specified *observation point* $\boldsymbol{r}$ at time t. However, it must be emphasized that the laws of mechanics (Newton's laws) apply to material particles or fluid elements, not to points of observation, so one must reformulate the equations of motion so that they apply to the Eulerian variables. The equations of motion in continuum mechanics are invariably expressed as partial differential equations for fields (such as the density of a fluid or height of a string at a given location and time). Thus, we are dealing with the classical dynamics of fields. We will now discuss the flow of fluids, primarily from an Eulerian perspective.

2 Introduction to fluid mechanics

Fluid flows are all around us: the air through our nostrils, tea stirred in a cup, water down a river and charged particles in the ionosphere. The flow of fluids can be fascinating to watch. It is also an interesting branch of physics to which many of the best scientists from the early days of Leonardo da Vinci, Isaac Newton, Daniel Bernoulli and Leonhard Euler have contributed. Fluid dynamics finds application in numerous areas: flight of airplanes and birds, weather prediction, blood flow in the heart and blood vessels, waves on the beach, ocean currents and tsunamis, flows in the molten metallic core of the Earth, controlled nuclear fusion in a tokamak, jet engines in rockets, motion of charged particles in the solar corona and astrophysical jets, accretion disks around active galactic nuclei, formation of clouds, melting of glaciers, climate change, sea level rise, traffic flow, building pumps and dams, etc. Fluid flows can range from regular and predictable (laminar) to seemingly disorganized and unpredictable (turbulent) while displaying remarkable patterns.

We believe[2] we know the macroscopic physical laws governing fluid motion. In

[2]Unlike in the application of Newton's laws to a pair of point particles or a rigid body, there are signif-

the absence of dissipation, they are the local conservation laws of mass, momentum and energy along with a thermodynamic equation of state. The resulting equation for the flow velocity in 'ideal' (dissipationless) flow goes back to the work of Euler (1757). In the presence of dissipation (viscosity, thermal conductivity, etc.), local conservation of mass continues to hold although the ideal momentum and energy equations are modified (in the simplest possible way) using empirical macroscopic laws of Newton and Fourier governing diffusion of momentum and heat to arrive at the equations for viscous flow. The corresponding equation for the flow velocity was introduced by Claude-Louis Navier (1822) and George Gabriel Stokes (1845). It is important to bear in mind that these equations of macroscopic fluid mechanics were postulated based on empirical observations, macroscopic conservation laws and the principles of minimality and simplicity rather than by a direct application of Newton's second law to individual molecules. In fact, these equations were proposed well before the molecular structure of matter was established. What is more, although we now know the laws of molecular dynamics accurately, it has not been possible to rigorously deduce the equations of fluid mechanics from them[3]. In this framework, the equations of fluid mechanics have to be validated by comparing their predictions[4] with macroscopic experimental measurements and observations. Fortunately, in many cases where such comparisons have been possible, there is evidence in favor of the fluid equations. However, there are situations where one needs to modify them (e.g., to account for a nonlinear stress-strain relation or the polymeric structure of constituent molecules) or abandon them (e.g., when one is interested in phenomena on molecular length scales).

3 Fluid element, local thermal equilibrium and dynamical fields

In a fluid description, we do not follow the microscopic positions and velocities of individual molecules. We focus instead on macroscopic fluid variables such as velocity, pressure, density, energy and temperature that we assign to a *fluid element* by averaging over it. By a fluid element (sometimes called a material element), we mean a sufficiently large collection of molecules so that concepts such as 'volume occupied' make sense and yet small in extent compared to the macroscopic length scales of phenomena we wish to describe. Thus, quantities such as the density and velocity will be assumed not to vary appreciably over a fluid element. For example, we could divide a bucket with about 10^{23} molecules into 10^3 fluid elements, each containing

icant approximations, imprecise notions of averaging and plausible assumptions involved in arriving at the equations governing macroscopic fluid motion.

[3] Well after their formulation, some of these macroscopic fluid equations (especially for dilute gases, but not for liquids) have been shown (by L Boltzmann, S Chapman, D Enskog and others) to follow from the molecular kinetic theory of gases through a coarse-graining procedure based on some plausible assumptions and approximations. In this chapter, we will introduce the equations of fluid mechanics from a macroscopic viewpoint and make no attempt to derive them from kinetic theory.

[4] As in the rest of continuum mechanics, the evolution equations of fluid dynamics are partial differential equations. However, these equations are nonlinear and despite much progress since the time of Euler, Navier and Stokes, it is still a challenge to calculate (even with the best of computers) many features of commonly occurring flows.

10^{20} molecules. Thus, we model a fluid as a continuum system with an infinite number of degrees of freedom[5]. The fluid description applies to phenomena on length scales large compared to the typical mean free path between collisions of molecules. On shorter length scales, the fluid description breaks down[6], though Boltzmann's kinetic theory of molecules applies.

A flowing fluid is generally *not* in global thermal equilibrium. What this means is that it may not be possible to assign a common temperature to all parts of a fluid, and heat could be transported between parts of a fluid. Nevertheless, collisions between molecules typically establish local thermodynamic equilibrium so that we may assign a local absolute temperature T, pressure p and density ρ to fluid elements, satisfying an equation of state (such as that of an ideal gas[7] $p = \rho RT/\mu$). Sometimes, it is convenient to replace some of these thermodynamic state variables with specific entropy s (entropy S per unit mass) or specific internal energy ϵ (energy per unit mass) or specific volume $v = 1/\rho$. Each of these quantities could vary from one fluid element to another and also with time. From an Eulerian standpoint, at each location $\boldsymbol{r}$ in a fluid at time t, we have the dynamical fields of density $\rho(\boldsymbol{r}, t)$, pressure $p(\boldsymbol{r}, t)$, specific entropy $s(\boldsymbol{r}, t)$, temperature $T(\boldsymbol{r}, t)$, etc. In addition to these scalar fields, we have the velocity vector field $\boldsymbol{v}(\boldsymbol{r}, t)$ that is instantaneously tangent to the flow at each point $\boldsymbol{r}$.

4 Fluid statics: aero- or hydrostatics

Before considering fluid flows in more detail, we briefly remark upon the special situation that prevails when the fluid is not in motion in the frame considered. This is usually called hydrostatics or sometimes aerostatics (if one wishes to emphasize that the density is inhomogeneous). In fluid static equilibrium, each fluid element is at rest due to a balance between surface and body forces. Surface forces are those that act on the element across its boundary due to material just outside the surface. The most common body force is gravity, which acts over the whole volume of the fluid element. To obtain the equations of hydrostatic equilibrium, we consider a small fluid element of mass $\delta m = \rho\, \delta V$ occupying a volume δV. The external body force such as gravity acting on the fluid element is $\boldsymbol{f} \delta V$ where $\boldsymbol{f}$ is the body force per unit volume ($\boldsymbol{f} = \rho \boldsymbol{g}$ for gravity, where $\boldsymbol{g}$ is the acceleration vector due to gravity). In addition, we have the surface force due to the pressure exerted on the fluid element by the fluid surrounding the element. To calculate this, assume the fluid element is an infinitesimal cuboid with sides of length dx, dy and dz.

As shown in Fig. 1, the net pressure force in the $\hat{x}$ direction is the product of the area $dydz$ and pressure difference between the left and right faces: $\delta F_x \approx -\frac{\partial p}{\partial x} dx \times dydz$. The negative sign is because pressure tends to compress the element and the

[5]To specify the pattern of a flow we must, among other things, specify the fluid velocity at each of the infinitely many points in the container.

[6]In going from a molecular description to a fluid description, we replace sums over individual molecules by integrals over the region occupied by the fluid, with fluid elements roughly playing the role of infinitesimal integration elements. A system with a very large but finite number of molecular degrees of freedom is approximated by a continuum system with infinitely many degrees of freedom.

[7]Here $R = 8.314$ Joules per Kelvin per mole is the universal gas constant and μ the molar mass, 12 grams per mole for Carbon-12

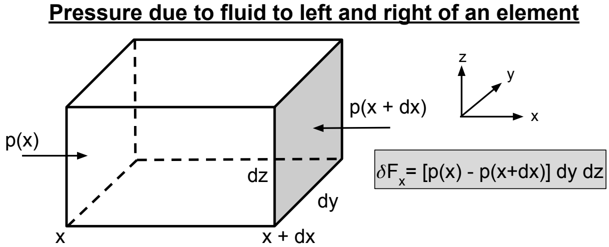


Figure 1: Horizontal force on a fluid element due to material to the left and right.

net force is leftward if p on the right face is larger than on the left face. Thus, the total pressure force on the fluid element is

$$\delta \boldsymbol{F}_{\text{pressure}} = \delta F_x \hat{x} + \delta F_y \hat{y} + \delta F_z \hat{z} = -(\boldsymbol{\nabla} p)\, \delta V. \tag{1}$$

For the element to be in static equilibrium, we must have

$$\boldsymbol{f} - \boldsymbol{\nabla} p = 0 \quad \text{or} \quad \boldsymbol{\nabla} p = \rho \boldsymbol{g}. \tag{2}$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ are the body forces per unit volume and mass respectively. This is one equation for two unknown functions, the pressure and density. It is usually supplemented by an 'equation of state' relating pressure to density. For an incompressible liquid, ρ can often be assumed to be a constant. For an ideal gas at a fixed temperature T, the equation of state is $p = \rho RT/\mu$ where μ is the molar mass and R the universal gas constant. This is usually written as Boyle's law $(p/p_o) = (\rho/\rho_o)$ where p_o is the pressure at a reference density ρ_o. If the pressure and density variations are at constant entropy (reversible adiabatic process) rather than constant temperature, the corresponding formula is $(p/p_o) = (\rho/\rho_o)^\gamma$ where the adiabatic index $\gamma = C_p/C_v$ is the ratio of heat capacities at constant pressure and volume. □

Example: Atmospheric pressure. For example, let us find the density and pressure as a function of height z in the atmosphere, assuming it is in aerostatic equilibrium and treating the temperature and acceleration due to gravity as independent of height. The force balance equation reduces to

$$\frac{\partial p}{\partial z} = -g\rho(z) \quad \text{or} \quad \frac{dp}{p} = -\frac{g\rho_o}{p_o} \quad \Rightarrow p(z) = p(0) e^{-\rho_o g z/p_o}. \tag{3}$$

Thus, the pressure and density decrease exponentially with height if we ignore the temperature and gravity variations. Prob. **??** treats this aerostatic situation with the isentropic equation of state $p \propto \rho^\gamma$, which is more realistic. □

A frequently encountered circumstance is one where the body force field per unit mass is the (negative) gradient of a potential $\boldsymbol{g} = -\boldsymbol{\nabla}\varphi$. Such a force is called *conservative*. Then $\boldsymbol{\nabla} p = -\rho \boldsymbol{\nabla} \varphi$. If, moreover, the density is a constant, we have $\boldsymbol{\nabla}(\frac{p}{\rho} + \varphi) = 0$. So $p/\rho + \varphi$ must be a constant. In particular, an equipotential surface must also be a surface of constant pressure (an isobar). For example, the free surface of a liquid is an isobar (pressure equal to atmospheric pressure), and hence must also be an equipotential surface within these approximations.

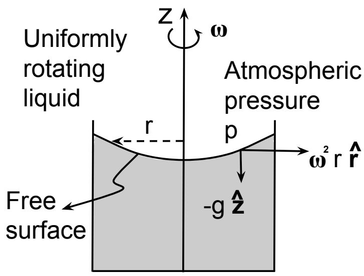


Figure 2: Parabolic free surface of a uniformly rotating liquid.

Example: Free surface of rotating liquid. Let us apply (2) to determine the shape of the free surface of a liquid that is rotated at a constant angular velocity $\omega\hat{z}$ in a bucket (cf. Fig. 2). After some time, the surface of the liquid is found to reach an equilibrium shape. In a corotating frame, the body forces per unit mass are gravity $-g\hat{z}$ and the centrifugal force $r\omega^2\hat{r}$ where we use cylindrical coordinates r, θ, z. Thus, the body force per unit mass is the negative gradient of the effective potential $\varphi = gz - \frac{1}{2}r^2\omega^2$. Once the liquid settles into equilibrium, $p/\rho + gz - \frac{1}{2}\omega^2 r^2$ is a constant. On the free surface, the pressure is constant, equal to atmospheric pressure. So the equation for the free surface $gz - \frac{1}{2}\omega^2 r^2 =$ constant, describes a paraboloid obtained by rotating the parabola $gz - \frac{1}{2}\omega^2 x^2 =$ constant, about the z axis. □

5 Flow visualization: streamlines, pathlines and streaklines

In fluid mechanics, when we speak of the velocity of a flow, we are referring not to the random thermal motions of individual molecules, but to the velocity of the overall flow. The latter is smoother since an average over molecules in each fluid element has been performed to arrive at the flow velocity field.

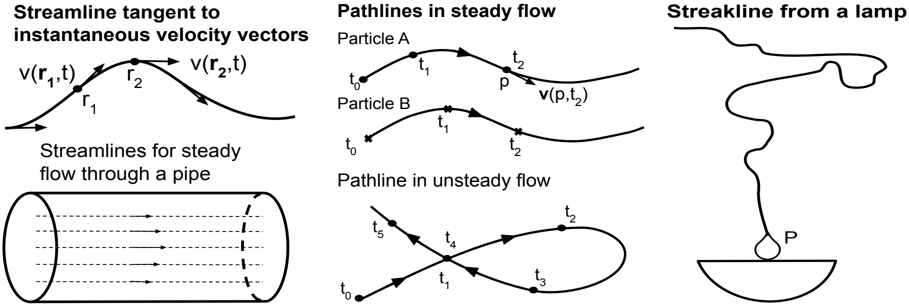
If the velocity vector field at every point of observation is independent of time, we say the velocity field is steady, $\boldsymbol{v}(\boldsymbol{r}, t) = \boldsymbol{v}(\boldsymbol{r})$. More generally, we will say that a fluid flow is steady if the velocity, density, pressure, temperature, specific entropy, etc., are independent of time at every point of observation. To aid in the visualization of a flow we define the concepts of streamlines, streaklines and pathlines. All three coincide for a steady flow, though not in general. For steady flow, they are the 'field lines' or integral curves of the velocity vector field, i.e., curves that are everywhere tangent to $\boldsymbol{v}(\boldsymbol{r})$ (see Fig. 3a). They are the trajectories of test particles moving in the steady flow, i.e., solutions of the ODEs and initial conditions

$$\frac{d\boldsymbol{r}}{ds} = \boldsymbol{v}(\boldsymbol{r}(s)) \quad \text{and} \quad \boldsymbol{r}(s_o) = \boldsymbol{r}_o. \tag{4}$$

Here, s is the parameter along the integral curve, it is the time that parametrizes the trajectory of the test particle moving in the steady flow. If we write these in Cartesian components $\boldsymbol{r}(s) = (x(s), y(s), z(s))$ and $\boldsymbol{v}(\boldsymbol{r}) = (v_x(\boldsymbol{r}), v_y(\boldsymbol{r}), v_z(\boldsymbol{r}))$, then the ODEs for field lines become

$$\frac{dx}{ds} = v_x, \quad \frac{dy}{ds} = v_y \quad \text{and} \quad \frac{dz}{ds} = v_z \quad \text{or} \quad \frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z} = ds. \tag{5}$$

Streamlines. More generally, consider a possibly nonsteady flow. Streamlines at the observation time t_o are defined as the integral curves of the velocity field $\boldsymbol{v}(\boldsymbol{r}, t_o)$. The streamline through any point of observation P with position vector $\boldsymbol{r}(P)$ at a given time t_o is tangent to the velocity vector $\boldsymbol{v}(\boldsymbol{r}(P), t_o)$. At a given instant of time, streamlines cannot intersect. Since the flow may not be steady, the streamlines will in general change with time. Streamlines of the velocity field are analogous to the field lines of a (generally time-dependent) electric or magnetic field. In particular, for a divergence-free ($\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$) flow, streamlines cannot emerge or spread out from a point or region, just as magnetic field lines cannot. A flow that is spatio-temporally regular is called laminar. An example is the slow, steady flow of water through a pipe, where streamlines are parallel as in Fig. 3a.



(a) (b) (c)

Figure 3: (a) Streamlines encode the instantaneous flow pattern. (b) Pathline of a speck of sawdust as it is carried by a flow. (c) Caricature of a streakline in the air above a lamp's burning wick at the point P. The burning wick introduces particles of soot into the air, which are carried by the air flow. The curve along which the soot lies at a given time is the instantaneous streakline. A burning incense stick also produces a streakline if we ignore the slow movement of the point of injection (reduction in length of the stick as it burns).

Streamlines have information on the current flow. For example, we could draw the streamlines of the monsoon winds over the Indian peninsula at the onset of the South-West monsoon on June 5, 2012. These streamlines changed with time and partly reversed direction during the 'receding' North-East monsoon in November 2012.

Pathlines are the trajectories of individual fluid particles. For example, if we introduced a small speck of saw dust[8] (which reflects light) into the fluid and took a movie of its trajectory, we would get its pathline (see Fig. 3b). At any point P along a pathline, it is tangent to the velocity vector at P at the time the particle passed through P. Pathlines can intersect themselves or even retrace themselves, for instance if a fluid particle goes round and round in a container. Two pathlines can intersect if the point of intersection corresponds to a different time on each of the two trajectories. For

[8]Leonardo da Vinci (1452-1519) suspended fine sawdust in water and observed the motion of the saw dust as it was carried by the flow. By contrast, pollen grains were used by Robert Brown (1827) to indirectly reveal the random thermal motion of molecules under a microscope.

example, two different dust particles may pass through the same point in a room on two different days.

Streaklines. Suppose a small quantity of dye is continuously injected into a fluid flow at a fixed point of injection P. The dye is so chosen that the dye particles do not diffuse in the fluid. Rather, a dye particle tends to stick to the first fluid particle it encounters and flows along with it. So the dye released at time t sticks to the fluid particle that passes through P at time t and is then carried by that particle. The resulting highlighted curve is the *streakline* through P as illustrated in Fig. 3c. So at a given time of observation $t_{\rm obs}$, a streakline is the locus of all current locations of particles that passed through P at some time $t \leq t_{\rm obs}$ in the past. Unlike streamlines, streaklines provide information on the history of the flow. Streaklines for a given flow are governed by three quantities: the point of injection P, the time of observation $t_{\rm obs}$ and the time when the injection of dye began t_i. Such a streakline always begins at P and extends to a point determined by t_i when injection began. In practice, streaklines get blurred by diffusion of the dye in the fluid, however they are reasonably sharp for a time short compared to the diffusion time scale. A streakline cannot self-intersect.

6 Material derivative

In the Eulerian description of fluid motion, we are interested in the time development of various fluid dynamical variables such as velocity, pressure, density and temperature at a given point of observation $\boldsymbol{r} = (x, y, z)$ in the container. This is reasonable if we are interested in predicting the weather changes at the point of observation over the course of time. For instance, the change in density at a fixed location is $\frac{\partial \rho(\boldsymbol{r})}{\partial t}$. However, different fluid particles will arrive at the point $\boldsymbol{r}$ as time passes. It is also of interest to know how the corresponding dynamical variables evolve, not at a fixed location but for a fixed small fluid element, as in a Lagrangian description. This is especially important since the dynamical laws of mechanics apply directly to the fluid particles, not to the point of observation. So, we may ask how a variable changes along the flow, so that the observer is always attached to a fixed fluid element (or 'material element') and travels along its pathline. For instance, the change in density of a fluid element in a small time dt as it moves from location $\boldsymbol{r}$ to $\boldsymbol{r} + d\boldsymbol{r}$ is

$$d\rho = \rho(\boldsymbol{r} + d\boldsymbol{r}, t + dt) - \rho(\boldsymbol{r}, t) \approx d\boldsymbol{r} \cdot \boldsymbol{\nabla} \rho + \frac{\partial \rho}{\partial t}\, dt. \qquad (6)$$

We divide by dt, take the limit $dt \to 0$ and observe that $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$ is the velocity of the fluid at the point $\boldsymbol{r}$ at time t. Thus, the instantaneous rate of change of density of a fluid element that is located at $\boldsymbol{r}$ at time t is

$$\frac{D\rho}{Dt} \equiv \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \rho = \left(\partial_t + v_x \partial_x + v_y \partial_y + v_z \partial_z\right) \rho. \qquad (7)$$

$\frac{D}{Dt} = \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}$ is called the material[9] (also total, substantial, convective) derivative. It can be used to express the rate of change of a physical quantity (velocity,

[9]The adjectives *material* or *substantial* are meant to convey that D/Dt is a rate of change computed while moving with the material or substance.

pressure, temperature, etc.) associated to a fixed fluid element, i.e., along the flow specified by the velocity field $\boldsymbol{v}$. This formula for the material derivative bears a resemblance to the rigid body formula relating the time derivatives of a vector relative to the lab and corotating frames: $\left(\frac{d\boldsymbol{A}}{dt}\right)_{\rm lab} = \left(\frac{d\boldsymbol{A}}{dt}\right)_{\rm rot} + \boldsymbol{\Omega} \times \boldsymbol{A}$. A quantity f (could be a scalar or a vector) is said to be conserved along the flow or dragged by the flow if its material derivative vanishes $\frac{Df}{Dt} = 0$.

Since $\frac{D}{Dt}$ is a first order partial differential operator, Leibniz's product rule of differentiation holds for scalar functions f, g: $\frac{D(fg)}{Dt} = f\frac{Dg}{Dt} + \frac{Df}{Dt}g$. Similarly, for a scalar f and vector field $\boldsymbol{w}$, we check that the Leibniz rule holds

$$\frac{D(f\boldsymbol{w})}{Dt} = \frac{Df}{Dt}\,\boldsymbol{w} + f\frac{D\boldsymbol{w}}{Dt}. \tag{8}$$

7 Compressibility, incompressibility and divergence of velocity field

We define a flow to be incompressible if the volume occupied by any fixed fluid element[10] (not necessarily small) remains constant in time although its shape may change. This is approximately true for water flowing in a hose pipe. Generally, liquids tend to be incompressible, they offer a large opposing force to attempts to change volume. Gases are more compressible, and high speed flows in gases tend to be compressible. However, the same material (like air) under different conditions may behave differently, depending on the speed of the flow in comparison to the speed of sound, as we will explain later in this section.

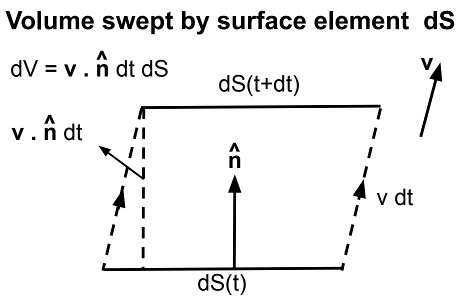


Figure 4: Surface element $d\boldsymbol{S} = \hat{\boldsymbol{n}}\,dS$ is carried by a flow $\boldsymbol{v}$ over a time dt sweeping out a volume $dV = \boldsymbol{v}\cdot\hat{\boldsymbol{n}}\,dt\,dS$. The figure shows a side view of the volume.

To clarify the idea of incompressibility, we ask how the volume V of a region Ω occupied by a fluid changes with time[11], i.e., we seek an expression for $\frac{dV}{dt}$. Suppose Ω is bounded by a surface $S = \partial\Omega$ with outward area element $d\boldsymbol{S}$ and outward unit normal $\hat{\boldsymbol{n}}$ such that $d\boldsymbol{S} = \hat{\boldsymbol{n}}\,dS$. In a small time dt, the region Ω changes by a

[10]By a fixed fluid element we mean a fixed collection of molecules. One can think of them as being surrounded by an imaginary impermeable membrane that instantaneously assumes the shape of the region they occupy.

[11]Here, $\frac{dV}{dt}$ is not the material derivative in the strict sense of Sect. 6, since V is not a local field. However, it is similar in spirit as it is the rate of change of volume following the flow.

movement of its bounding surface[12] in the direction of $\boldsymbol{v}$. At a point $\boldsymbol{r}$ on $\partial\Omega$, the surface element dS moves out a perpendicular distance $\boldsymbol{v}\cdot\hat{\boldsymbol{n}}\,dt$ where $\boldsymbol{v}$ is the fluid velocity at the point $\boldsymbol{r}$ (see Fig. 4). Thus, the change in volume $dV(dS)$ due to the area element dS moving out a bit is $\boldsymbol{v}\cdot\hat{\boldsymbol{n}}\,dt\,dS$. To include the contributions of all area elements, we integrate over the entire bounding surface to arrive at

$$\frac{dV}{dt} = \int_S \boldsymbol{v}\cdot\hat{\boldsymbol{n}}\,dS = \int_\Omega \boldsymbol{\nabla}\cdot\boldsymbol{v}\,d\boldsymbol{r}. \tag{9}$$

The last equality uses Gauss' divergence theorem to transform the surface integral into a volume integral. Since this is true for a fluid parcel of any volume (above molecular sizes), let us specialize to a small fluid element Ω (so that $\boldsymbol{\nabla}\cdot\boldsymbol{v}$ is roughly constant over its extent) at location $\boldsymbol{r}$ having volume δV. Then,

$$\frac{d\delta V}{dt} = \frac{D\,\delta V}{Dt} \approx (\boldsymbol{\nabla}\cdot\boldsymbol{v})\,(\delta V) \quad \text{or} \quad \boldsymbol{\nabla}\cdot\boldsymbol{v} = \lim_{V\to 0}\frac{1}{V}\frac{dV}{dt} = \lim_{V\to 0}\frac{d\log V}{dt}. \tag{10}$$

So the divergence of the velocity field is the fractional rate of change of volume of a small fluid element.

A flow is incompressible if each fluid element maintains its volume during the flow, i.e., $\frac{dV}{dt} = 0$ for all V (above molecular scales). It follows that a flow is incompressible iff the velocity field is divergence-free: $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$.

Examples. A simple example of an incompressible flow is one where the density of the fluid is the same everywhere and at all times. In fact, if the density ρ is a constant, then the volume of an element is a fixed multiple $(1/\rho)$ of its mass. However, the mass of a *material* element is conserved, so its volume must remain constant. A more general example of an incompressible flow is one where the density of a given fluid element is constant in time, though different fluid elements may have different densities. This happens for horizontal flows in the atmosphere, where the density is stratified by height though the flow is horizontal. Note that the same fluid (e.g., air) under different conditions may exhibit incompressible and compressible flows. The study of compressible flows is usually termed gas dynamics or aerodynamics, while the study of incompressible flows is often termed hydrodynamics.

Compressibility and bulk modulus. Incompressibility means the volume of a fluid element does not change irrespective of the pressure applied across its surface. A measure of the compressibility[13] of a flow is the *compressibility* $\kappa = -\frac{1}{V}\frac{\partial V}{\partial p}$. The negative sign ensures that $\kappa \geq 0$, since pressure tends to decrease volume in most materials. Thus, $\kappa \to 0$ in an incompressible flow.

The reciprocal of compressibility is called the bulk modulus K. Since the mass of a fluid element is conserved, incompressibility may be taken to mean that the density

[12] We neglect the infinitesimal change in the area dS of the surface element due to the flow. The change in volume due to such a change is of second order in infinitesimals. The surface area of a material element can change even in incompressible flow.

[13] Intuitively, compressibility measures how much the volume of a fluid element decreases in response to a unit increase in applied pressure. To obtain a nontrivial limit as $V \to 0$, we divide by the volume V of the fluid element to arrive at the local (intensive) variable κ.

does not change with applied pressure. Indeed, since $\rho \propto 1/V$, we may write the compressibility[14] also as $\kappa = \frac{1}{\rho}\frac{\partial \rho}{\partial p}$. □

Relation to speed of sound. Intuitively, a sound wave is a wave of compression and expansion. As we will learn in Sect. **??**, a sound wave propagates changes in density and travels at the speed c_s where $c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$ (for flow with constant specific entropy s). Evidently, c_s grows as the compressibility κ decreases. Solids tend to be less compressible than gases. As a consequence, sound propagates faster in steel than in air and we can hear an approaching train on a railway track earlier than it is heard through the air. If the flow velocity $|\boldsymbol{v}|$ is small compared to the speed of sound c_s, then the flow can usually be approximated as incompressible[15]. In fact, we may regard a strictly incompressible flow as one where the speed of sound is infinite. Crudely, any attempt by the flow to alter the density of a fluid element is immediately wiped out since sound travels much faster than the flow and irons out the change. □

Incompressibility in 2d: stream function. The condition for a vector field on the x-y plane to be incompressible can be solved in terms of a scalar *stream function* $\psi(x, y)$. Indeed, suppose $\boldsymbol{v} = (u(x, y), v(x, y), 0)$, then $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ becomes the condition $u_x + v_y = 0$, where subscripts denote partial derivatives. Now, if

$$u = \psi_y \quad \text{and} \quad v = -\psi_x, \tag{11}$$

then the incompressibility condition is identically satisfied. In 3d vector notation, we can regard $\psi(x, y)\hat{z}$ as a vector potential for the incompressible velocity field: $\boldsymbol{v} = \boldsymbol{\nabla} \times (\psi\hat{z})$, which is then automatically divergence-free. This is similar to how the solenoidal magnetic field is expressed in terms of a vector potential $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ in electrodynamics. □

To sum up, we introduced the idea of incompressibility via the divergence of $\boldsymbol{v}$ and then discussed the physical meaning of compressibility in terms of the density ρ. Pleasantly, the divergence-free condition $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ may be expressed in terms of the material derivative of ρ via the continuity equation, as we will see in Sect. 8.

8 Local conservation of mass: continuity equation

The total mass of fluid in a given fluid element remains constant in time, since material does not enter or leave the element. Consider a small fluid element of volume δV in the vicinity of the point $\boldsymbol{r}$ where the fluid density is $\rho(\boldsymbol{r})$ at time t. Then the mass of the fluid element is $\delta m = \rho\, \delta V$. The material derivative of δm must vanish. Using the Leibniz rule (8) and (10) we get for any small δV,

$$0 = \frac{D\,\delta m}{Dt} = \frac{D(\rho\,\delta V)}{Dt} = \frac{D\rho}{Dt}\,\delta V + \rho\,\frac{D\,\delta V}{Dt} = \left(\frac{D\rho}{Dt} + \rho \boldsymbol{\nabla} \cdot \boldsymbol{v}\right)\delta V. \tag{12}$$

[14] In evaluating this partial derivative using the thermodynamic equation of state (see Sect. 10), a third variable such as temperature or entropy is held fixed. So one has slightly different notions of compressibility depending on what is held fixed.

[15] The Mach number $M = |\boldsymbol{v}|/c_s$ (which could depend on location and time) is a way of quantifying this. The Mach number is zero in incompressible flow. Flow in regions where $M < 1$ is called subsonic while it is supersonic where $M > 1$.

Thus, we arrive at the *continuity equation* expressing conservation of mass

$$\frac{D\rho}{Dt} + \rho \, \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0. \tag{13}$$

We immediately see that if the density is constant along the flow ($\frac{D\rho}{Dt} = 0$), then the flow is divergence-free ($\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$) and incompressible. Expanding the material derivative, we get

$$\frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \rho + \rho \, \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0. \tag{14}$$

In particular, if $\rho = \rho_0$ is constant in both time and space, then the flow must be incompressible. On the other hand, if the flow is incompressible, i.e., $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$, then the density must be constant along the flow, $\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\rho = 0$. We say the density is advected or transported by an incompressible flow.

Combining the last two terms in (14), the continuity equation can be written in local conservation form:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0. \tag{15}$$

We say that ρ is the locally conserved mass density and $\rho \boldsymbol{v}$ is the corresponding mass current density. The continuity equation says that the rate of change of density at a point is balanced by the divergence of the mass current density. We may also write (15) in integral form, by integrating over a region Ω that is *fixed* in space (does not move with the flow) and applying Gauss' divergence theorem:

$$\int_\Omega \frac{\partial \rho}{\partial t} \, d\boldsymbol{r} + \int_\Omega \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) \, d\boldsymbol{r} = 0 \quad \text{or} \quad \frac{d}{dt} \int_\Omega \rho \, d\boldsymbol{r} + \int_{S = \partial\Omega} \rho \boldsymbol{v} \cdot d\boldsymbol{S} = 0. \tag{16}$$

The 1^{st} term is the rate of increase of mass inside a fixed volume Ω. The 2^{nd} gives the outward flux of mass across the boundary S. So mass is neither created nor destroyed: it can only move around *continuously*, hence the name 'continuity' equation. If Ω is the entire flow domain, then the first term is the rate of increase of mass of the fluid as a whole, which must vanish provided the mass flux across the boundary is zero.

9 Euler equation for inviscid flow

An inviscid (sometimes called ideal) fluid flow is one where no resistance is offered to changes in shape that are not accompanied by a change in volume. We will elaborate on this shortly. In particular, ideal fluids assume the shape of the container; they lack a rigidity of form. This means that in an ideal flow, the force acting on a material element (anywhere in the fluid) across its surface, due to the material outside, is everywhere normal to the surface. Tangential surface forces tend to *shear* the element and change its shape without affecting its volume. On the other hand, normal surface forces tend to compress or expand[16] the element and thereby change its volume. The inward directed normal surface force per unit area is called pressure p. So in

[16] Normal surface forces that tend to expand an element are called tensile stresses, as is the case in an elastic rod that is being stretched. Tensile stresses correspond to a *negative* pressure.

an inviscid flow, tangential or shearing stresses vanish irrespective of the location and orientation of the surface. In viscous flows, tangential forces typically arise between layers of fluid in relative motion. Thus, tangential forces are absent in hydrostatics.

Stress tensor[17]**.** In general, forces need not be either normal or tangential to surfaces[18] in the fluid, and they could vary in magnitude and direction with location. The stress tensor is a quantity that encodes the force per unit area acting across an element of surface. Let $\hat{\boldsymbol{n}}\,\delta S$ be a small surface element of area δS, with unit normal $\hat{\boldsymbol{n}}$, centered at $\boldsymbol{r}$. Let $\boldsymbol{F}(\hat{\boldsymbol{n}}\,\delta S, \boldsymbol{r})$ be the force that acts across the surface, its magnitude must be proportional to the area δS. Precisely, it is the force on the material on the side to which $\hat{\boldsymbol{n}}$ points, due to the material on the other side, as shown in Fig. 5. In general, $\boldsymbol{F}$ and $\hat{\boldsymbol{n}}$ point in different directions and are related by a linear transformation, the transformation of stress. If we choose to write all vectors in some basis, e.g., resolve them according to Cartesian components, then this linear relation may be written as

$$\boldsymbol{F}_i(\hat{\boldsymbol{n}}\,\delta S, \boldsymbol{r}) = \sum\nolimits_j T_{ij}(\boldsymbol{r}) n_j \,\delta S. \tag{17}$$

The 3×3 matrix $T_{ij}(\boldsymbol{r})$ is called the stress tensor field. It depends only on the location $\boldsymbol{r}$ and not on the surface or $\hat{\boldsymbol{n}}$. By choosing a surface whose normal $\hat{\boldsymbol{n}}$ points in the $j^{\rm th}$ direction, we see that T_{ij} is then the $i^{\rm th}$ component of the force acting on the material towards the $j^{\rm th}$ direction of a surface of unit area whose normal points in the $j^{\rm th}$ direction. Alternatively, suppose δS is a small surface with normal $\hat{\boldsymbol{n}}$, then $\sum_j T_{ij} n_j (\delta S)$ is the $i^{\rm th}$ component of the force acting on the material on the side to which the normal $\hat{\boldsymbol{n}}$ points.

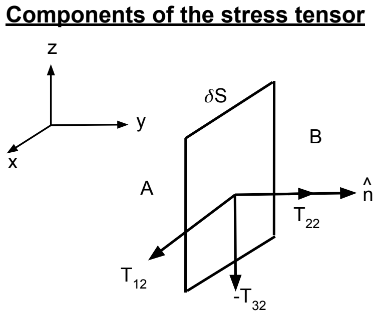


Figure 5: Components of the force due to fluid A on fluid B across a small surface with unit normal $\hat{\boldsymbol{n}}$ which here points along $\hat{y}$. T_{32} is the third component of the force on the material located on the second direction of the surface.

Example: stress tensor in hydrostatics and inviscid flow. By definition, hydrostatic pressure acts normal to any surface. So consider a small cuboid with axes along Cartesian axes. It follows that $T_{ij} = 0$ for $i \neq j$, as there are no tangential stresses. Moreover, $T_{33} = p$ since the force across the top surface (whose normal points along

[17] Here, we introduce the stress tensor in general, not necessarily for inviscid flow.

[18] These may be external or, more frequently, hypothetical internal surfaces in the fluid.

$\hat{z}$) due to the fluid below, is $p\hat{z}$. We get the same answer by considering the bottom surface. Proceeding in this way, $T_{ij} = p\delta_{ij}$. This formula for the stress tensor due to hydrostatic pressure is independent of basis: multiples of the identity matrix have the same components in any basis.

More generally, the absence of tangential stresses in an inviscid flow irrespective of orientation of surfaces implies that the stress tensor is diagonal in every basis, and must therefore be proportional to the identity: $T_{ij} = p\delta_{ij}$. □

Euler equation. To derive the equation of motion for an inviscid flow, consider a small fluid element of mass $\delta m = \rho\,\delta V$ occupying a volume δV and having instantaneous velocity $\boldsymbol{v}$. Let us write Newton's $2^{\rm nd}$ law for this fluid element. The change in its velocity in a time dt as it is displaced from $\boldsymbol{r}$ to $\boldsymbol{r} + d\boldsymbol{r}$ is

$$d\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r} + d\boldsymbol{r}, t + dt) - \boldsymbol{v}(\boldsymbol{r}, t) \approx \frac{\partial \boldsymbol{v}}{\partial t} dt + (d\boldsymbol{r} \cdot \boldsymbol{\nabla})\boldsymbol{v}. \tag{18}$$

Dividing by dt, letting $dt \to 0$ and noting that $d\boldsymbol{r}/dt = \boldsymbol{v}$, we obtain its acceleration: $D\boldsymbol{v}/Dt \equiv \partial \boldsymbol{v}/\partial t + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}$. The material derivative $D\boldsymbol{v}/Dt$ differs from the partial derivative by the quadratically nonlinear 'advection' term $(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}$. By Newton's $2^{\rm nd}$ law, the force acting on the element must equal $\rho\,\delta V\,\frac{D\boldsymbol{v}}{Dt}$.

We consider two sorts of forces acting on the fluid element. There can be an external force field such as gravity (called a body force) acting on the fluid. It may be expressed as $\boldsymbol{f}\delta V$ where $\boldsymbol{f}(\boldsymbol{r})$ is the body force per unit volume (e.g., $\boldsymbol{f} = \rho \boldsymbol{g}$ where $\boldsymbol{g}$ is the acceleration due to gravity). In addition, we have the surface force due to the pressure exerted on the element by the fluid surrounding the element. To calculate this, assume the fluid element is a cuboid with sides dx, dy, dz. The net pressure force in the $\hat{x}$ direction is the product of the area $dy\,dz$ and pressure differential between the two faces: $\delta F_x = -\frac{\partial p}{\partial x} dx \times dy\,dz$. The $-$ sign arises because if p is greater on the right face of the element compared to the left face, then the net force would be leftward. Thus, the total surface force[19] on the fluid element is $\delta \boldsymbol{F} = -(\boldsymbol{\nabla} p)\delta V$. Thus, Newton's $2^{\rm nd}$ law for the fluid element reads

$$\rho\,\delta V\,\frac{D\boldsymbol{v}}{Dt} = -(\boldsymbol{\nabla} p)\,\delta V + \boldsymbol{f}\,\delta V. \tag{20}$$

Dividing by δV, we get Euler's celebrated equation of motion[20] for an inviscid fluid.

[19]More generally, the force due to pressure across the surface $\partial(\delta V)$ of the element is

$$\delta \boldsymbol{F}_{\rm surface} = -\int_{\partial(\delta V)} p\hat{\boldsymbol{n}}\,dS = -\int_{\delta V} \boldsymbol{\nabla} p\,dV \approx -\boldsymbol{\nabla} p\,\delta V. \tag{19}$$

We have used a corollary of Gauss' divergence theorem to convert the surface integral to a volume integral and taken $\boldsymbol{\nabla} p$ to be constant over the small volume δV. The minus sign is because $\hat{\boldsymbol{n}}$ is the outward-pointing normal.

[20]The Euler equation can be written in terms of the stress tensor $T_{ij} = p\delta_{ij}$

$$\partial_t v_i + v_j \partial_j v_i = -\frac{1}{\rho}\partial_j T_{ij} + \frac{1}{\rho} f_i \quad \text{in Cartesian components.} \tag{21}$$

The equation may be generalized to viscous flows by including tangential stresses in T_{ij} (see Sect. **??**). Here, repeated indices are summed and no distinction is made between upper and lower indices.

It must be considered in conjunction with the continuity equation (13)

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \frac{\boldsymbol{f}}{\rho} \quad \text{and} \quad \frac{D\rho}{Dt} + \rho\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0. \tag{22}$$

Notice that the Euler equation is quadratically nonlinear in $\boldsymbol{v}$ due to the $\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}$ advection term. This makes it difficult to solve but also allows it to describe a wide variety of ideal flows.

A vector identity allows us to write the advection term in terms of the *vorticity* $\boldsymbol{w} = \boldsymbol{\nabla}\times\boldsymbol{v}$ and a gradient term:

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{w}\times\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}p - \frac{1}{2}\boldsymbol{\nabla}v^2 + \frac{\boldsymbol{f}}{\rho} \tag{23}$$

Here $\boldsymbol{w}\times\boldsymbol{v}$ is called the vortex force per unit mass or Lamb vector. We will have more to say about vorticity in Sect. **??**.

The Euler and continuity equations are first order in time derivatives of $\boldsymbol{v}$ and ρ. So we need to specify the initial values $\rho(\boldsymbol{r},0)$ and $\boldsymbol{v}(\boldsymbol{r},0)$, to be able to evolve them forward in time[21]. However, these are still only four evolution equations for five unknown functions (density, pressure and three components of the velocity field). In particular, we have not specified how the pressure evolves in time. We will address this question for adiabatic flow in Sect. 10. Here, we deal with the slightly simpler case of incompressible constant density flow.

Pressure for constant density flow. If $\rho(\boldsymbol{r},t) = \bar\rho$ is a constant in space and time, then the continuity equation (14) implies $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$. Taking the divergence of the Euler equation (22) (in the absence of external body forces), the time derivative term is eliminated leaving us with a nondynamical 'constraint' equation

$$\boldsymbol{\nabla}^2 p = -\bar\rho\,\boldsymbol{\nabla}\cdot(\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}). \tag{24}$$

If we view the RHS as a source, this is Poisson's equation[22] for p. It can be solved with suitable boundary conditions, say using Green's function for the Laplace operator. For decaying BCs, we have

$$p(\boldsymbol{r},t) = \frac{\bar\rho}{4\pi}\int \frac{\boldsymbol{\nabla}\cdot(\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v})(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d\boldsymbol{r}'. \tag{25}$$

Thus, for constant density, we have been able to eliminate the pressure from the Euler equation, which becomes an evolution equation for $\boldsymbol{v}$ alone. We say that in constant

[21] In addition, we need to impose suitable *boundary conditions*. The Euler and continuity equations are first order in space derivatives, and we may impose conditions on the boundary values of $\boldsymbol{v}$ and ρ. On fixed impenetrable boundaries, the normal component $\boldsymbol{v}\cdot\hat{\boldsymbol{n}}$ must vanish. In the absence of viscosity, the tangential component of $\boldsymbol{v}$ is unconstrained on boundaries. In unbounded regions, we typically have decaying BCs: $\boldsymbol{v}\to 0$ and $\rho\to\rho_0$ as $|\boldsymbol{r}|\to\infty$.

[22] In electrostatics, when the electric field is expressed in terms of an electrostatic potential ($\boldsymbol{E} = -\boldsymbol{\nabla}\phi$), Gauss' law $\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho/\epsilon_0$ leads to Poisson's equation $\boldsymbol{\nabla}^2\phi = -\rho/\epsilon_0$, where $\rho(\boldsymbol{r})$ is the electric charge density. The solution involves the Coulomb potential, which is essentially the Green function of the Laplace operator: $\phi = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d\boldsymbol{r}'$.

density flow, the pressure is not dynamical. It does not obey an independent evolution equation but is determined by the instantaneous velocity distribution. See Prob. **??** for the case of incompressible flow with variable density. □

10 Ideal adiabatic flow: entropy advection and equation of state

As pointed out below Eq. (22), the Euler and continuity equations (22) are generally an underdetermined system: they do not tell us how the pressure evolves. To understand how the pressure evolves, we need to broaden our physical perspective. Recall from Sect. 3 that a fluid can usually be considered to be in local thermal equilibrium. This means there is a local temperature field $T(\boldsymbol{r}, t)$ that, along with the pressure and density, satisfies an equation of state ($p = \rho k_b T/m$ for an ideal gas with molecular mass m.). To find the remaining dynamical equation, it is fruitful to ask how the conjugate variable to T, i.e., the entropy evolves. For a dissipationless flow, it is physically reasonable to suppose that the entropy of a fluid element remains constant in time, just as its mass does. In other words, there is no entropy production or heat exchanged between fluid elements. Such a flow is called adiabatic.

Dynamics of specific entropy. Now consider a small fluid element of volume δV and let s denote the specific entropy field (entropy per unit mass). Then the entropy of the fluid element is $\rho\, s\, \delta V$. If this is conserved as the element moves around, then its material derivative must vanish.

Using the Leibniz rule and (10), we get

$$\frac{D(\rho s \delta V)}{Dt} = \rho s (\boldsymbol{\nabla} \cdot \boldsymbol{v}) \delta V + \delta V \frac{D(\rho s)}{Dt} = 0 \quad \text{or} \quad \partial_t(\rho s) + \boldsymbol{\nabla} \cdot (\rho s \boldsymbol{v}) = 0. \tag{26}$$

In other words, the entropy per unit volume ρs is locally conserved[23] with the corresponding entropy current given by $\rho s \boldsymbol{v}$. Using the continuity equation (14), the adiabaticity of the flow implies that s is advected by $\boldsymbol{v}$:

$$\partial_t s + \boldsymbol{v} \cdot \boldsymbol{\nabla} s = 0. \tag{27}$$

This is our third evolution equation. The pressure is then determined by the equation of state, which may be regarded as a relation among s, p and ρ. For instance, for an ideal gas with constant specific heat ratio $\gamma = c_p/c_v$, the equation of state is

$$s = c_v \log \left(\frac{p/\bar{p}}{(\rho/\bar{\rho})^\gamma} \right) \tag{28}$$

for some reference values $\bar{p}$ and $\bar{\rho}$ (see Prob. **??**).

Internal energy or pressure equation. We may also combine this equation of state (28), the entropy advection equation (27) and the continuity equation (14) to derive an evolution equation for pressure (see Prob. **??**):

$$\left(\frac{p}{\gamma - 1} \right)_t + p \boldsymbol{\nabla} \cdot \boldsymbol{v} + \boldsymbol{\nabla} \cdot \left(\frac{p \boldsymbol{v}}{\gamma - 1} \right) = 0. \tag{29}$$

[23] Integrating over the flow domain and assuming the entropy flux across the boundary vanishes, we arrive at the global conservation of entropy $\frac{d}{dt} \int \rho\, s\, d\boldsymbol{r} = 0$.

This is called the internal energy equation since $p/(\gamma - 1)$ will be interpreted as the internal energy density of an ideal gas (see Sect. **??**).

Homentropic and barotropic flow. Homentropic flow is a situation where the entropy advection equation (27) can be eliminated. Here, the specific entropy $s = s_0$ is independent of both space and time and (27) is identically satisfied. Moreover, the equation of state then becomes a relation between ρ and p. In general, a relation between ρ and p is called a barotropic relation. For example[24], for homentropic flow of an ideal gas with adiabatic index γ, the barotropic relation can be written as $(p/p_0) = (\rho/\rho_0)^\gamma$ for some reference values p_0 and ρ_0. For barotropic flow, pressure $p(\boldsymbol{r}, t)$ is determined by the instantaneous density $\rho(\boldsymbol{r}, t)$ and we do not need to supplement the continuity and Euler equations by a third evolution equation.

Remark. Note that for $\gamma = 1$, the barotropic relation for homentropic flow of an ideal gas becomes $p = (p_0/\rho_0)\rho$ where p_0/ρ_0 is a constant. Comparing with the ideal gas law $p = (k_b T/m)\rho$, we infer that the temperature in such a flow, $T = mp_0/k_b\rho_0$, is spatially constant and independent of time. Thus, such a flow must be isothermal. However, not all isothermal flows arise this way. A gas with $\gamma = c_p/c_v \neq 1$ can display an isothermal flow. □

An important consequence of a barotropic relation expressing $\rho = \rho(p)$ is that the pressure term on the RHS of the Euler equation (23) can be expressed as a gradient:

$$\frac{\boldsymbol{\nabla} p}{\rho} = \boldsymbol{\nabla} h \quad \text{where} \quad h(p) = \int_{p_0}^{p} \frac{dp'}{\rho(p')} \quad \text{so} \quad \boldsymbol{\nabla} h = h'(p) \boldsymbol{\nabla} p = \frac{1}{\rho} \boldsymbol{\nabla} p. \tag{30}$$

For barotropic (homentropic) flow of an ideal gas,

$$\frac{\boldsymbol{\nabla} p}{\rho} = \gamma \frac{p_0}{\rho_0} \left(\frac{\rho}{\rho_0}\right)^{\gamma - 1} \frac{\boldsymbol{\nabla} \rho}{\rho} = \frac{\gamma}{\gamma - 1} \boldsymbol{\nabla} \left(\frac{p}{\rho}\right) \quad \Rightarrow \quad h = \frac{\gamma}{\gamma - 1} \left(\frac{p}{\rho}\right). \tag{31}$$

Here, $h(\rho)$ is called the specific enthalpy or enthalpy per unit mass[25]. If, in addition, the body force per unit mass can be expressed as a gradient, $\boldsymbol{f}/\rho = -\boldsymbol{\nabla}\varphi$ [i.e., body force is conservative], then the RHS of Euler's equation (22) becomes a gradient:

$$\frac{D\boldsymbol{v}}{Dt} = \partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = -\boldsymbol{\nabla}(h + \varphi). \tag{32}$$

What is more, using the identity (**??**) to write the advection term in terms of the vortex force, the Euler equation becomes

$$\partial_t \boldsymbol{v} + \boldsymbol{w} \times \boldsymbol{v} = -\boldsymbol{\nabla}(\sigma + \varphi) \quad \text{where} \quad \sigma = h + \frac{1}{2} v^2. \tag{33}$$

Here, σ is called the *stagnation enthalpy*, it reduces to the enthalpy at a stagnation point (i.e., one where $\boldsymbol{v} = 0$).

[24] Another example of barotropic flow is the isothermal inviscid compressible flow of an ideal gas. The barotropic relation is $p = \rho k_b T/m$ where T is the constant temperature and m the mass of a molecule. In this case, the role of specific enthalpy is played by the specific Gibbs free energy $g(\rho) = (k_b T/m) \log(\rho/\rho_0)$ which is determined up to a constant by $\boldsymbol{\nabla} g = (\boldsymbol{\nabla} p)/\rho$.

[25] The first law of thermodynamics $dU = TdS - pdV$, when written in terms of enthalpy $H = U + pV$ instead of internal energy U, becomes $dH = TdS + Vdp$. For an isentropic process $dS = 0$, so $dh = dp/\rho$. Here $V = M/\rho$ is the volume, M the mass of fluid, $h = H/M$ the enthalpy per unit mass, T absolute temperature and S the entropy.

11 Bernoulli's equation

Bernoulli's principle for steady flow. Recall from Sect. 5, that a fluid flow is steady if $\boldsymbol{v}, \rho, p$, etc., are not explicitly dependent on time. In its simplest form, Bernoulli's principle concerns a drop in pressure along a streamline in places where a steady constant density flow speeds up. Euler's equation (33) for a steady homentropic flow with specific enthalpy $h(\rho)$ and body force potential φ is

$$\boldsymbol{v} \times \boldsymbol{w} = \boldsymbol{\nabla} \left(\frac{1}{2} \boldsymbol{v}^2 + h + \varphi \right) \quad \text{where} \quad \boldsymbol{w} = \boldsymbol{\nabla} \times \boldsymbol{v}. \tag{34}$$

For example, $\varphi = gz$ for the gravitational body force, where z is the vertical height and g the magnitude of the acceleration due to gravity. The left member is orthogonal to $\boldsymbol{v}$, so upon taking the dot product with the velocity field, we get Bernoulli's equation:

$$\boldsymbol{v} \cdot \boldsymbol{\nabla} \mathcal{B} = 0 \quad \text{where} \quad \mathcal{B} = \frac{1}{2} \boldsymbol{v}^2 + h + \varphi. \tag{35}$$

Thus, the component of the gradient of the Bernoulli specific energy $\mathcal{B}$ along the velocity vector field is zero. If $\boldsymbol{r}(s)$ is a streamline[26], then Bernoulli's equation becomes

$$\frac{d\boldsymbol{r}}{ds} \cdot \boldsymbol{\nabla} \mathcal{B} = 0 \quad \text{or} \quad \frac{d\mathcal{B}(\boldsymbol{r}(s))}{ds} = 0. \tag{36}$$

So in steady flow, $\mathcal{B} = \frac{1}{2}\boldsymbol{v}^2 + h + \varphi$ is constant along streamlines. Note that $\mathcal{B}$ will, in general, take different values for different streamlines. Now recall that the enthalpy per unit mass is $h = \varepsilon + \frac{p}{\rho}$ where ε is the internal energy per unit mass, p the pressure and ρ the density. Thus, for steady homentropic inviscid flow subject to a conservative body force, Bernoulli's equation says that along streamlines,

$$\mathcal{B} = \frac{1}{2} v^2 + \varepsilon + \frac{p}{\rho} + \varphi \quad \text{is conserved.} \tag{37}$$

If the flow is incompressible, then ρ is constant along the flow and, in particular, along streamlines of the steady flow. Suppose the internal energy density of the fluid is also constant along the flow. Then we find that $\frac{1}{2}\rho v^2 + p + \rho\varphi$ is constant along streamlines. If in addition, the body force potential φ does not vary along the streamline (as for horizontal streamlines in a vertical gravitational field), then $\frac{1}{2}\rho v^2 + p$ is constant along streamlines. In other words, in regions of high pressure along a streamline, the fluid speed must be low and vice-versa. Such a situation is approximately encountered in laminar flow through a cylindrical pipe of varying cross section. On account of mass conservation, the water speeds up in regions where there is a constriction in the pipe. At such constrictions, the pressure drops, as can be demonstrated by comparing the pressure with atmospheric pressure (a lower pressure supports a shorter vertical column of water against atmospheric pressure).

[26] A streamline $\boldsymbol{r}(s)$ is an integral curve of the velocity vector field: $\frac{d\boldsymbol{r}}{ds} = \boldsymbol{v}(\boldsymbol{r}(s))$. Here, s is a parameter along the streamline.

Bernoulli equation for unsteady flow. There is a version of the Bernoulli equation (35) that applies to unsteady flows, though in the restricted context of barotropic potential flow ($\boldsymbol{v} = \boldsymbol{\nabla}\phi$). Potential flow is irrotational $\boldsymbol{w} = \boldsymbol{\nabla} \times \boldsymbol{v} = 0$, so the vortex force vanishes and the Euler equation (33) for barotropic flow subject to a body force derived from a potential ($\boldsymbol{f}/\rho = -\boldsymbol{\nabla}\varphi$) becomes

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{\nabla} h - \boldsymbol{\nabla}\left(\frac{1}{2}v^2\right) - \boldsymbol{\nabla}\varphi \quad \text{or} \quad \boldsymbol{\nabla}\left(h + \frac{\partial \phi}{\partial t} + \frac{1}{2}v^2 + \varphi\right) = 0. \tag{38}$$

The quantity in parentheses must be independent of location but could depend on time. Thus, we arrive at the unsteady Bernoulli equation for barotropic potential flow:

$$\frac{\partial \phi}{\partial t} + h + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \varphi = B(t). \tag{39}$$

The simplest case is that of constant density, were $h = p/\rho$. Unlike Bernoulli's equation (35) for steady flow, (39) holds throughout the fluid and is not associated with streamlines. The unsteady Bernoulli equation may also be interpreted as an evolution equation for the velocity potential ϕ. It can also be used to eliminate the pressure p in favor of the velocity potential when computing the force due to pressure[27] on a body immersed in a fluid.

[27] If S is a surface with fluid to one side of it, then the force on the surface due to fluid pressure is given by $\int_S p\,\hat{\boldsymbol{n}}\,dA$ where dA is the area element and $\hat{\boldsymbol{n}}$ is the unit normal pointing away from the fluid.