Thermal Physics, Autumn 2019 CMI
Problem set 3
Due by the beginning of lecture on Tuesday, Sep 10, 2019
First law, Adiabatic model for atmospheric temperature and pressure profile

1. ⟨**5**⟩ Use the first law to show that the difference in heat capacities is given by

$$C_p - C_V = \left(\left(\frac{\partial U}{\partial V}\right)_T + p\right)\left(\frac{\partial V}{\partial T}\right)_p. \tag{1}$$

2. ⟨**2**⟩ Now suppose a gas satisfies the ideal equation of state $pV = nRT$ and the 'Caloric condition'. Use the above to evaluate the difference $C_p - C_V$ for such an ideal gas.

3. ⟨**20**⟩ It is known that temperature, density and pressure all decrease with height z in the atmosphere. We wish to find the temperature and pressure gradients in a simple model for the atmosphere. We assume that the air is an ideal gas satisfying the ideal gas law $pV = nRT$. Since air is a poor conductor we will assume that parcels of air move without much heat exchange with surroundings (i.e. adiabatically), so we may assume that any two among p, V, T satisfy the adiabatic relation. We will assume a steady state where layers of air in the atmosphere are in equilibrium due to a balance of their weight by the upward pressure gradient.

   (a) ⟨**3**⟩ Find the differential condition for mechanical equilibrium of a thin layer of air of surface area A at height z and thickness dz. Denote the pressure and density by $p(z)$ and $\rho(z)$.

   (b) ⟨**3**⟩ Show that the variation of pressure satisfies the equation (μ is average molar mass for air and g the acceleration due to gravity)

$$\frac{dp}{p} = -\frac{\mu g}{RT} dz \tag{2}$$

   (c) ⟨**5**⟩ Show that the temperature gradient dT/dz is a constant κ (assuming g is constant). Find an analytic formula for κ in terms of the adiabatic index γ.

   (d) ⟨**3**⟩ Find the numerical value of the temperature gradient by taking reasonable values for the constants.

   (e) ⟨**3**⟩ Find a differential equation for the variation of pressure with height in the form $dp/dz = f(p, z)$, assuming temperature on the Earth's surface ($z = 0$) is T_0.

   (f) ⟨**3**⟩ Find a formula for $p(z)$, assuming the pressure at $z = 0$ is p_0