Problem set 2: Classical Mechanics: 2^{nd} module

Refresher course on classical mechanics and electromagnetism
Sponsored by the three Indian Academies of Sciences & conducted at
Sri Dharmasthala Manjunatheshwara College, Ujire, Karnataka, Dec 8-20, 2014

1. The first law of thermodynamics says that the change in internal energy of a gas is equal to the heat supplied to the gas minus the work done by the gas. For infinitesimal reversible changes, $dU = TdS - PdV$. Here dU is the increase in internal energy, P the pressure, dV the increase in volume dS the increase in entropy and T the absolute temperature.

   (a) What are the independent variables that U depends on?

   (b) Helmholtz free energy may be introduced via the formula $F = U - TS$. Find the independent variables that F depends on.

   (c) Express the pressure and entropy in terms of the Helmholtz free energy.

   (d) Write a formula for Helmholtz free energy as a Legendre transform of the internal energy. Indicate which variable to extremize in and give the condition for an extremum.

2. Find a smooth convex function $L(v)$ of a real variable v whose Legendre transform $H(p) = \text{ext}_v(pv - L(v))$ is the same function as L. In other words, $H(x) = L(x)$ for any $x \in \mathbb{R}$. You may first try to guess such a function using physical knowledge. But you must also formulate the above condition as an equation and try to solve it to determine such a function. Give a physical interpretation of the resulting function $H(p)$.

3. Find the *unequal* time p.b. $\{q(0), q(t)\}$ for a free particle of mass m moving on a line.

4. Consider a particle moving on the plane $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(x, y)$. x, y, p_x, p_y are the usual coordinates and momenta on phase space satisfying canonical Poisson bracket relations. Define the dynamical variables (plane polars) $r(x, y) = \sqrt{x^2 + y^2}$ and $\theta(x, y) = \arctan(y/x)$. Recall that the Lagrangian in terms of polar coordinates is $\tilde{L}(r, \theta, \dot{r}, \dot{\theta}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \tilde{V}(r, \theta)$. We wish to compute the p.b. among polar coordinates and their conjugate momenta using the definition $\{f, g\} = f_x g_{p_x} - f_{p_x} g_x + f_y g_{p_y} - f_{p_y} g_y$. Here subscripts denote partial derivatives.

   (a) Express the conjugate momenta p_r, p_θ as functions on phase space (i.e., in terms of x, y, p_x, p_y). Show that you get

   $$p_r = \frac{x}{r(x, y)} p_x + \frac{y}{r(x, y)} p_y \quad \text{and} \quad p_\theta = x p_y - y p_x. \tag{1}$$

   (b) Find the partial derivatives (denoted by subscripts) $r_x, r_y, \theta_x, \theta_y$. Show that you get

   $$r_x = \frac{x}{r}, \quad r_y = \frac{y}{r}, \quad \theta_x = -\frac{y}{r^2}, \quad \theta_y = \frac{x}{r^2}. \tag{2}$$

   (c) Find the partial derivatives of r, θ with respect to p_x and p_y: $r_{p_x}, r_{p_y}, \theta_{p_x}, \theta_{p_y}$.

   (d) Find the partial derivatives of p_r, p_θ with respect to x, y, p_x, p_y. You must give 8 formulae, two of which must be shown to be

   $$(p_r)_x = \frac{p_x}{r} - \frac{x^2}{r^3} p_x - \frac{xy}{r^3} p_y \quad \text{and} \quad (p_r)_y = \frac{p_y}{r} - \frac{y^2}{r^3} p_y - \frac{xy}{r^3} p_x. \tag{3}$$

(e) Find the 6 (after accounting for anti-symmetry) p.b. among polar coordinates and momenta (i) $\{r, \theta\}$, (ii) $\{r, p_r\}$, (iii) $\{r, p_\theta\}$, (iv) $\{\theta, p_\theta\}$, (v) $\{\theta, p_r\}$ and (vi) $\{p_r, p_\theta\}$.

(f) Comment on the result.

5. Consider a free particle moving on the half line $q > 0$ with Lagrangian $L(q, \dot{q}) = \frac{1}{2}m\dot{q}^2$. Suppose we make the change of coordinate to $Q = q^2$.

   (a) Find the new Lagrangian $\tilde{L}(Q, \dot{Q})$.

   (b) Find the momentum P conjugate to Q. Express P as a function of Q and $\dot{Q}$ and as a function of q and p.

   (c) Find the Poisson bracket $\{Q, P\}$ and compare with $\{q, p\}$.

6. Angular momentum Poisson brackets from $\{r_i, p_j\} = \delta_{ij}$. We place all indices down-stairs in this problem, and sum repeated indices.

   (a) Define the Levi-Civita symbol ϵ_{ijk} for $1 \leq i, j, k \leq 3$ by the condition that it is antisymmetric under interchange of any pair of *neighboring* indices along with the 'initial' condition $\epsilon_{123} = 1$. Show that it is anti-symmetric under interchange of any pair of indices, (not necessarily neighbors).

   (b) Give the values of all the components of the ϵ symbol. How many components are there in all?

   (c) From $\vec{L} = \vec{r} \times \vec{p}$ write the three components of angular momentum L_x, L_y, L_z in terms of x, y, z, p_x, p_y, p_z and show that they may be summarized by the formula $L_i = \epsilon_{ijk} r_j p_k$. Repeated indices are summed.

   (d) Use the properties of the Poisson bracket and the identity

   $$\sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}. \tag{4}$$

   to show that

   $$\{L_i, L_j\} = r_i p_j - r_j p_i. \tag{5}$$

   For uniformity of notation, begin by taking $L_i = \epsilon_{ikl} r_k p_l$ and $L_j = \epsilon_{jmn} r_m p_n$

   (e) Use the above formula for L_i to show that $\epsilon_{ijk} L_k = r_i p_j - r_j p_i$.

   (f) Conclude from the last two questions that

   $$\{L_i, L_j\} = \epsilon_{ijk} L_k \tag{6}$$

   Compare this with the 3 formulae derived in lecture: $\{L_x, L_y\} = L_z$ and cyclic permutations thereof. Do they agree?

   (g) Use the above results to show that

   $$\{\{L_i, L_j\}, L_k\} = \delta_{ik} L_j - \delta_{jk} L_i. \tag{7}$$

   (h) Show that the components of angular momentum satisfy the Jacobi identity

   $$\{\{L_i, L_j\}, L_k\} + \{\{L_j, L_k\}, L_i\} + \{\{L_k, L_i\}, L_j\} = 0. \tag{8}$$