SINGLE POINT SESHADRI CONSTANTS ON RATIONAL SURFACES

KRISHNA HANUMANTHU AND BRIAN HARBOURNE

ABSTRACT. Motivated by a similar result of Dumnicki, Küronya, Maclean and Szemberg under a slightly stronger hypothesis, we exhibit irrational single-point Seshadri constants on a rational surface X obtained by blowing up very general points of $\mathbb{P}^2_{\mathbb{C}}$, assuming only that all prime divisors on X of negative self-intersection are smooth rational curves C with $C^2 = -1$. (This assumption is a consequence of the SHGH Conjecture, but it is weaker than assuming the full conjecture.)

1. Introduction

In spite of the many constraints now known on the possible values of Seshadri constants (see for example [2, 3, 4, 5]), the longstanding question of whether Seshadri constants on surfaces (defined below) can ever be irrational remains open. In the case of a surface X obtained as the blow up $\pi : X \to \mathbb{P}^2$ of the complex projective plane $\mathbb{P}^2$ at very general points $p_1, \cdots, p_s \in \mathbb{P}^2$, recent work of Dumnicki, Küronya, Maclean and Szemberg, [1, Main Theorem], shows for $s \geq 9$ that the SHGH Conjecture implies that certain ample divisors L on X have irrational Seshadri constants $\varepsilon(X, L, x)$ when x is a very general point of X. In this note we show that less is needed to obtain this conclusion, namely one merely has to assume that prime divisors C on the blow up Y of X at x with $C^2 < 0$ satisfy $C^2 = C \cdot K_Y = -1$. This assumption is itself a consequence of the SHGH Conjecture but it is not known to be equivalent to the full SHGH Conjecture, and it leads to a conceptually simpler proof than the one obtained in [1]. It also leads us to raise the question if an even weaker assumption, viz., Nagata's Conjecture, suffices to draw the same conclusion.

2. Main result

We recall some standard facts. Given a point x on a smooth projective surface S and an ample divisor L, the Seshadri constant $\varepsilon(S, L, x)$ is defined to be

$$\varepsilon(S, L, x) = \inf_C \frac{L \cdot C}{\mathrm{mult}_x(C)},$$

where the infimum is taken over all curves C containing x. Alternatively, let $\pi : Y \to S$ be the blow up of S at x with exceptional curve E. Then $\varepsilon = \varepsilon(S, L, x)$ is the supremum of all real t such that $\pi^*(L) - tE$ is nef and hence $(\pi^*(L) - \varepsilon E)^2 \geq 0$. It follows that $\varepsilon(S, L, x) \leq \sqrt{L^2}$. If $\varepsilon(S, L, x) < \sqrt{L^2}$, one says that $\varepsilon(S, L, x)$ is submaximal, in which case it is well known that there exists a reduced and irreducible curve C on S passing

Date: November 20, 2017.
2010 *Mathematics Subject Classification.* Primary 14C20; Secondary 14H50, 14J26.
First author was partially supported by a grant from Infosys Foundation.

through x such that $\varepsilon = \varepsilon(S, L, x) = \frac{L \cdot C}{\mathrm{mult}_x(C)}$ (i.e., such that $(\pi^*(L) - \varepsilon E) \cdot \tilde{C} = 0$, where $\tilde{C} \subset Y$ is the strict transform of C). Such a curve C is called a *Seshadri curve* for L at x. Since $\varepsilon = \varepsilon(S, L, x) < \sqrt{L^2}$ implies $(\pi^*(L) - \varepsilon E)^2 > 0$, it follows by the Hodge index theorem that $\tilde{C}^2 < 0$.

We will also need to refer to multi-point Seshadri constants. Given distinct points $p_1, \cdots, p_s$ on S and an ample divisor L, the multi-point Seshadri constant $\varepsilon(S, L, p_1, \cdots, p_s)$ is defined to be

$$\varepsilon(S, L, p_1, \cdots, p_s) = \inf_C \frac{L \cdot C}{\sum_i \mathrm{mult}_{p_i}(C)},$$

where the infimum is taken over all curves C containing at least one of the points p_i. Alternatively, let $\pi : Y \to S$ be the blow up of S at $p_1, \cdots, p_s$ with E_i being the exceptional curve for p_i. Then $\varepsilon = \varepsilon(S, L, p_1, \cdots, p_s)$ is the supremum of all real t such that $\pi^*(L) - t(E_1 + \cdots + E_s)$ is nef and hence $(\pi^*(L) - \varepsilon(E_1 + \cdots + E_s))^2 \geq 0$. If $0 < t < \varepsilon$, it is easy to see that $\pi^*(L) - t(E_1 + \cdots + E_s)$ is ample (since $F = (t/\varepsilon)(\pi^*(L) - \varepsilon(E_1 + \cdots + E_s))$ is nef and meets any nonnegative linear combination of the E_i positively, and $\pi^*(L) - t(E_1 + \cdots + E_s) = F + (1 - (t/\varepsilon))\pi^*(L))$. When the points p_i are very general, we will write $\varepsilon = \varepsilon(S, L, s)$ for $\varepsilon = \varepsilon(S, L, p_1, \cdots, p_s)$.

Our focus will be on surfaces $\pi : Y \to X \to \mathbb{P}^2$ where $X \to \mathbb{P}^2$ is obtained by blowing up very general points $p_1, \cdots, p_s$ on $\mathbb{P}^2$ and $Y \to X$ is the blow up of a very general point $x \in X$ with exceptional divisor E. So let $H = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and let E_i be the exceptional curve for each point p_i. Every divisor on Y is linearly equivalent to a unique integer linear combination $F = dH - mE - m_1E_1 - \cdots - m_sE_s$. (Since $Y \to X$ is an isomorphism away from x, we can regard the divisors H and E_i as also being on X. With this abuse of notation, every divisor on X is linearly equivalent to a unique integer linear combination $dH - m_1E_1 - \cdots - m_sE_s$.) Such a divisor F is in *standard* form if $m \geq m_1 \geq \cdots \geq m_s \geq 0$ and $d \geq m + m_1 + m_2$. An *exceptional* curve on X (or Y) is a reduced and irreducible rational curve C with $C^2 = -1$ (and hence $-K_X \cdot C = 1$, or $-K_Y \cdot C = 1$ respectively). If F is in standard form, then $F \cdot C \geq 0$ for all exceptional curves C on Y. (To see this, let $F = dH - mE - m_1E_1 - \cdots - m_sE_s$ be divisor on Y. If F is in standard form and if C is one of the exceptional curves $E, E_1, \cdots, E_s$ then clearly $F \cdot C \geq 0$. So suppose that C is different from $E, E_1, \cdots, E_s$. Note that F is in standard form if and only if F is a nonnegative linear integer combination of $H_0 = H$, $H_1 = H - E$, $H_2 = 2H - E - E_1$, $H_3 = 3H - E - E_1 - E_2, \cdots, H_{s+1} = 3H - E - E_1 - \cdots - E_s = -K_Y$. But H_i is nef for $i = 0, 1, 2$ and $H_i \cdot C \geq -K_Y \cdot C = 1$ for $i \geq 3$.)

The above definition of standard divisors also extends to divisors with coefficients in $\mathbb{Q}$ or $\mathbb{R}$. If F is a standard $\mathbb{Q}$-divisor, then for a suitable positive integer n, the $\mathbb{Z}$-divisor nF is standard. It follows that $F \cdot C \geq 0$ for all exceptional curves C on Y. If F is a standard $\mathbb{R}$-divisor, then F is the limit of a sequence of standard $\mathbb{Q}$-divisors. So again $F \cdot C \geq 0$ for all exceptional curves C on Y.

Proposition 2.1. *Let $s \geq 13$ be an integer with $s \neq 15, 16$. Let $X \to \mathbb{P}^2_{\mathbb{C}}$ be the blow up of $\mathbb{P}^2 = \mathbb{P}^2_{\mathbb{C}}$ at s very general points $p_1, \cdots, p_s$ and let $Y \to X$ be the blow up of X at a very general point $x \in X$. Suppose that every reduced and irreducible curve C on Y with $C^2 < 0$ is an exceptional curve. Then there exists an ample line bundle L on X such that the Seshadri constant $\varepsilon(X, L, x)$ is irrational for any very general point $x \in X$.*

Proof. Let $L = dH - E_1 - \cdots - E_s$ be a divisor on X with $4d - 3 \leq s < d^2$. By [6, Corollary] and [9, Theorem], L is ample. Let x be a very general point of X and let $\pi : Y \to X$ be the blow up at x with exceptional curve E.

We will show that there are no Seshadri curves for $L = dH - E_1 - \cdots - E_s$ at x if $4d-3 \leq s < d^2$. If there were a Seshadri curve C, then $\varepsilon = \varepsilon(X, L, x) < \sqrt{L^2} = \sqrt{d^2 - s}$, so $0 = (\pi^*(L) - \varepsilon E) \cdot \tilde{C} > (\pi^*(L) - \sqrt{d^2 - s}E) \cdot \tilde{C}$. Since $\tilde{C}^2 < 0$, by hypothesis we have that $\tilde{C}$ is an exceptional curve. But note that $\pi^*(L) - \sqrt{d^2 - s}E = dH - \sqrt{d^2 - s}E - E_1 - \cdots - E_s$ is in standard form: since $4d-3 \leq s$, we get $(d-2)^2 > d^2 - s$, so we have $d > \sqrt{d^2 - s} + 2$, and $d^2 > s$ so $d^2 - s \geq 1$, hence $\sqrt{d^2 - s} \geq 1$. It follows that $\pi^*(L) - \sqrt{d^2 - s}E$ meets all exceptional curves nonnegatively. Since $\tilde{C}^2 < 0$, by hypothesis we must have that $\tilde{C}$ is an exceptional curve. But then $(dH - \sqrt{d^2 - s}E - E_1 - \cdots - E_s) \cdot \tilde{C} < 0$ is not possible. Thus $\varepsilon(X, L, x)$ cannot be submaximal, so $\varepsilon(X, L, x) = \sqrt{L^2} = \sqrt{d^2 - s}$.

Alternatively, we can directly obtain the equality $\varepsilon(X, L, x) = \sqrt{L^2} = \sqrt{d^2 - s}$ when $4d-3 \leq s < d^2$, using the following argument suggested by the referee. It suffices to show that $\pi^\star L - \sqrt{d^2 - s}E$ is nef. Recall that a line bundle on a surface is nef if its intersection with every curve of negative self-intersection is nonnegative. Note that $\pi^\star L - \sqrt{d^2 - s}E$ is in standard form, as shown above. Hence it intersects all exceptional curves on Y nonnegatively. By assumption there are no other curves of negative self-intersection on Y. Thus $\pi^\star L - \sqrt{d^2 - s}E$ is nef and hence $\varepsilon(X, L, x) = \sqrt{L^2} = \sqrt{d^2 - s}$.

If $s \geq 13$ but $s \neq 15, 16$, we now show that d can be chosen so that $\sqrt{d^2 - s}$ is irrational. For $s = 13$ or 14, take $d = 4$; then $13 = 4d - 3 \leq s < d^2 = 16$, so $d^2 - s = 3$ or 2, hence $\sqrt{d^2 - s}$ is irrational. For $s \geq 17$, there is always a d with $4d - 3 \leq s \leq 6d - 10$, since $4d - 3 = 17$ for $d = 5$, while $4(d + 1) - 3 \leq (6d - 10) + 1$ for $d \geq 5$. Thus $(d - 3)^2 + 1 = d^2 - (6d - 10) \leq d^2 - s \leq d^2 - (4d - 3) = (d - 2)^2 - 1$, so $\sqrt{d^2 - s}$ again is irrational. □

Proposition 2.2. *Let $X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at s very general points where $s \in \{9, 10, 11, 12, 15, 16\}$. Let $Y \to X$ be the blow up of X at a very general point $x \in X$. Suppose that any irreducible and reduced curve on Y of negative self-intersection is exceptional. Then there is an ample line bundle L on X such that $\epsilon(X, L, x)$ is irrational.*

Proof. We consider different cases.

<u>$s = 9$</u>: Let $L = (3n + 1)H - n(E_1 + \cdots + E_9)$ for $n \geq 1$. Then $L^2 = 6n + 1 > 0$. Since $\varepsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), 9) = 1/3$, it follows that L is ample. Let $\pi : Y \to X$ be the blow up at a very general point $x \in X$ with exceptional curve E and let $\varepsilon = \varepsilon(X, L, x)$.

Note that $\pi^*(L) - \sqrt{6n + 1}E = (3n+1)H - n(E_1 + \cdots + E_9) - \sqrt{6n + 1}E$ is in standard form for $n \geq 7$ if we take the blow ups in the order $E_1, \cdots, E_9, E$, since $3n+1 > n+n+n$ and $n \geq \sqrt{6n + 1} \geq 0$. Now by the same argument used in the proof of Proposition 2.1, we conclude that $\pi^*(L) - \sqrt{6n + 1}E$ cannot meet any exceptional curve negatively. Hence $\epsilon(X, L, x)$ has to be maximal. Thus $\epsilon(X, L, x)$ is irrational provided $L^2 = 6n + 1$ is not a perfect square for some $n \geq 7$. This is the case for example for $n = 6m^2$ for any $m \geq 2$.

<u>$s = 10$</u>: Let $L = 10H - 3(E_1 + \cdots + E_{10})$. Then $L^2 = 10$. By hypothesis every curve on Y of negative self-intersection is exceptional. Clearly the same statement holds on X. Under this hypothesis, it is easy to show that the multi-point Seshadri constant $\varepsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), 10) = 1/\sqrt{10}$. It then follows that L is ample.

Note that $\pi^*(L) - \sqrt{10}E$ is in standard form (since $10 \geq \sqrt{10} + 6$). Hence by the same argument used above, we conclude that $\pi^*(L) - \sqrt{10}E$ cannot meet any exceptional curve negatively. Thus $\varepsilon(X, L, x) = \sqrt{10}$.

$\underline{s = 11}$: Let $L = 7H - 2(E_1 + \cdots + E_{11})$. The same argument as in the case $s = 10$ works to give $\varepsilon(X, L, x) = \sqrt{5}$.

$\underline{s = 12}$: Let $L = 11H - 3(E_1 + \cdots + E_{12})$. The same argument as in the case $s = 10$ works to give $\varepsilon(X, L, x) = \sqrt{13}$.

$\underline{s = 15}$: Let $L = 13H - 3(E_1 + \cdots + E_{15})$. The same argument as in the case $s = 10$ works to give $\varepsilon(X, L, x) = \sqrt{34}$.

$\underline{s = 16}$: Let $L = (4n + 1)H - n(E_1 + \cdots + E_{16})$. Then a similar argument as in the case $s = 9$ shows that L is ample and $\varepsilon(X, L, x)$ cannot be submaximal for any $n \geq 9$. So $\varepsilon(X, L, x) = \sqrt{L^2} = \sqrt{8n + 1}$. This is irrational for infinitely many $n \geq 9$. □

Remark 2.3. As is well known to experts [8], all single-point Seshadri constants on a blow up of $\mathbb{P}^2$ at $s \leq 8$ points are rational. For $s \leq 7$, this is because the subsemigroup of effective divisor classes of an 8 point blow up S of $\mathbb{P}^2$ is finitely generated, hence the nef cone is finite polyhedral with boundaries defined by negative effective classes and effective classes of self-intersection 0. The case of $s = 8$ is slightly more delicate since the subsemigroup of effective divisor classes of a 9 point blow up S of $\mathbb{P}^2$ need not be finitely generated, but it is generated by the exceptional curves and curves which occur as components of curves in the linear system $|-K_S|$, so again the nef cone has boundaries defined by negative effective classes and effective classes of self-intersection 0.

Combining Remark 2.3, Proposition 2.1 and Proposition 2.2, we obtain our main theorem.

Theorem 2.4. *Let $s \geq 0$ be an integer. Let $X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at s very general points $p_1, \cdots, p_s$ and let $Y \to X$ be the blow up of X at a very general point $x \in X$. Suppose that every reduced and irreducible curve C on Y with $C^2 < 0$ is an exceptional curve. Then there exists an ample line bundle L on X such that the Seshadri constant $\varepsilon(X, L, x)$ is irrational if and only if $s \geq 9$.*

Remark 2.5. In fact using the ideas in the proof of Proposition 2.1 and Proposition 2.2, we can get the following stronger assertion.

Let $s \geq 9$ be an integer. Consider the divisor $L_{d,n} = dH - n(E_1 + \cdots + E_s)$ on the blow up X of $\mathbb{P}^2$ at s very general points. Let $Y \to X$ be the blow up at a very general point. Suppose that every reduced and irreducible curve of negative self-intersection on Y is an exceptional curve. Then for infinitely many values of n, there exists a d such that $L_{d,n}$ is ample and the Seshadri constant $\varepsilon(X, L_{n,d}, x)$ is irrational for a very general point $x \in X$.

Our results depend only on assuming all negative curves are exceptional. A somewhat weaker result was conjectured by Nagata [7], namely for a blow up S of $\mathbb{P}^2$ at $s \geq 10$ very general points, if $dH - (m_1E_1 + \cdots + m_sE_s)$ is linearly equivalent to an effective divisor, then $d\sqrt{s} \geq \sum_i m_i$. This is equivalent to conjecturing that $F_0 = \sqrt{s}H - E_1 - \cdots - E_s$ is nef. Note for arbitrarily small $\delta > 0$ that $F_\delta = (\delta + \sqrt{s})H - E_1 - \cdots - E_s$ is rational and semi-effective (meaning that a positive integer multiple is linearly equivalent to an effective divisor, which follows since $F^2 > 0$). Thus if F_0 is not nef, then there is a prime divisor C with $C^2 < 0$ and $C \cdot F_0 < 0$. From this we see that the SHGH Conjecture implies Nagata's Conjecture. In fact, if C being a prime divisor with $C^2 < 0$ implies $C^2 = C \cdot K_S = -1$, then already Nagata's Conjecture is true. This is because if $C^2 < 0$ for a prime divisor C, then $C \cdot (\sqrt{s}H - E_1 - \cdots - E_s) \geq C \cdot (3H - E_1 - \cdots - E_s) \geq 1$.

Thus Nagata's Conjecture is weaker than the assumption we used. Note further that the Nagata Conjecture exhibits irrational multi-point Seshadri constants on $\mathbb{P}^2$, since it is equivalent to the statement that $\varepsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), s) = 1/\sqrt{s}$ for every $s \geq 10$. These remarks raise the following question.

Question 2.6. Is it possible to exhibit irrational single-point Seshadri constants on very general blow ups of $\mathbb{P}^2$ assuming only the Nagata Conjecture?

Acknowledgement: We thank Tomasz Szemberg for reading this paper and giving useful suggestions. We also thank the referee for giving an alternate argument in the proof of Proposition 2.1 and numerous other suggestions which improved the exposition.

References

[1] Dumnicki, M.; Küronya, A.; Maclean, C.; Szemberg, T.: *Rationality of Seshadri constants and the Segre-Harbourne-Gimigliano-Hirschowitz conjecture*, Adv. Math. 303 (2016), 1162-1170.

[2] Farnik, L.; Szemberg, T.; Szpond, J.; Tutaj-Gasińska, H.: *Restrictions on Seshadri constants on surfaces*, Taiwanese J. Math. 21 (2017), 27–41.

[3] Hanumanthu, K.; Mukhopadhyay, A.: *Multi-point Seshadri constants on ruled surfaces*, Proc. Amer. Math. Soc. 145 (2017), no. 12, 5145-5155.

[4] Harbourne, B.: *Seshadri constants and very ample divisors on algebraic surfaces*, J. Reine Angew. Math. 559 (2003) 115–122.

[5] Harbourne, B.; Roé, J.: *Discrete Behavior of Seshadri Constants on Surfaces*, J. Pure Appl. Algebra, 212 (2008), 616–627.

[6] Küchle, O.: *Ample line bundles on blown up surfaces*, Math. Ann. 304 (1996), 151–155.

[7] Nagata, M.: *On the 14-th problem of Hilbert*, Amer. J. Math. 81 (1959), 766–772.

[8] Sano, T.: *Seshadri constants on rational surfaces with anticanonical pencils*, J. Pure Appl. Algebra 218 (2014), no. 4, 602-617.

[9] Xu, G.: *Divisors on the blow-up of the projective plane*, Manuscripta Math. 86 (1995), 195-197.

Chennai Mathematical Institute, H1 SIPCOT IT Park, Siruseri, Kelambakkam 603103, India

E-mail address: krishna@cmi.ac.in

Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588, USA

E-mail address: brianharbourne@unl.edu