SESHADRI CONSTANTS ON HYPERELLIPTIC SURFACES

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Abstract. We prove new results on single point Seshadri constants for ample line bundles on hyperelliptic surfaces, motivated by the results in [10]. Given a hyperelliptic surface X and an ample line bundle L on X, we show that the least Seshadri constant $\varepsilon(L)$ of L is a rational number when X is not of type 6. We also prove new lower bounds for the Seshadri constant $\varepsilon(L,1)$ of L at a very general point.

1. Introduction

Let X be a smooth complex projective variety and let L be a line bundle on X. The Seshadri criterion [13, Theorem 7.1] for ampleness says that L is ample if and only if there exists a real number $\varepsilon > 0$ such that $L \cdot C \geq \varepsilon \cdot \text{mult}_x C$, where $x \in X$ is an arbitrary point and $C \subset X$ is any irreducible and reduced curve containing x (here $\text{mult}_x C$ denotes the multiplicity of the curve C at x). In other words, L is ample if and only if the infimum of the ratios $\frac{L\cdot C}{\text{mult}_x C}$, over all points x and all irreducible and reduced curves C passing through x, is positive. Using this idea Demailly [8] defined the notion of *Seshadri constants*. Given X, L as above, the *Seshadri constant* of L at $x \in X$ is defined as:

$$\varepsilon(X,L,x) := \inf_{x\in C} \frac{L\cdot C}{\text{mult}_x C},$$

where the infimum is taken over all irreducible and reduced curves passing through x. The Seshadri criterion for ampleness can now be stated simply as follows: L is ample if and only if $\varepsilon(X,L,x) > 0$ for all $x \in X$.

There are several interesting directions in which Seshadri constants are being studied. See [5] for a comprehensive survey. One of the important problems in the study of Seshadri constants is computing them or bounding them. In the present article we focus on this problem for hyperelliptic surfaces. In general, Seshadri constants are difficult to compute precisely and a lot of research has focussed on finding good lower and upper bounds.

Let X be a smooth complex projective surface and let L be an ample line bundle on X. It is easy to see that $\varepsilon(X,L,x) \leq \sqrt{L^2}$ for all x. One then defines

$$\varepsilon(X,L,1) := \sup_{x\in X} \varepsilon(X,L,x).$$

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It is known that $\varepsilon(X, L, 1)$ is attained at a *very general* point $x \in X$; see [17]. This means that $\varepsilon(X, L, 1) = \varepsilon(X, L, x)$ for all x outside a countable union of proper Zariski closed sets in X.

It is also known that if $\varepsilon(X, L, x) < \sqrt{L^2}$, then $\varepsilon(X, L, x) = \frac{L \cdot C}{\mathrm{mult}_x C}$ for some curve C ([6, Proposition 1.1]). If $\varepsilon(X, L, x) < \sqrt{L^2}$, then we say that it is *sub-maximal.* So sub-maximal Seshadri constants are always rational, while a maximal Seshadri constant is irrational if L^2 is not a square. However, no example is known of a triple (X, L, x) for which $\varepsilon(X, L, x) \notin \mathbb{Q}$.

At the other end of the interval, one defines

$$\varepsilon(X, L) := \inf_{x \in X} \varepsilon(X, L, x).$$

It is easy to see that $\varepsilon(X, L) > 0$ for ample L. In fact, $\varepsilon(X, L) \geq \frac{1}{n}$, if nL is very ample. Just like $\varepsilon(X, L, 1)$, it is known that $\varepsilon(X, L) = \varepsilon(X, L, x)$ for some $x \in X$ (see [17]). But unlike $\varepsilon(X, L, 1)$, which is attained at very general points, $\varepsilon(X, L)$ is attained at *special* points. In general, one has the following inequalities for any point $x \in X$:

$$0 < \varepsilon(X, L) \leq \varepsilon(X, L, x) \leq \varepsilon(X, L, 1) \leq \sqrt{L^2}.$$

Further, it follows from the previous paragraph that $\varepsilon(X, L) \in \mathbb{Q}$, except when L^2 is not a square and $\varepsilon(X, L, x) = \varepsilon(X, L) = \varepsilon(X, L, 1) = \sqrt{L^2}$ for all $x \in X$.

The above discussion leads to an interesting dynamic in the study of bounds on Seshadri constants. In many situations, the Seshadri constants of L at very general points may be expected to be maximal, i.e., equal to $\sqrt{L^2}$. On the other hand, the Seshadri constants at special points (and $\varepsilon(X, L)$) are expected to be sub-maximal and hence rational. This leads to two contrasting problems. On the one hand, the focus has been to find good lower bounds for $\varepsilon(X, L, 1)$ which are very close to the maximal value $\sqrt{L^2}$. The second problem is to calculate the Seshadri constants at special points and try to prove that $\varepsilon(X, L) \in \mathbb{Q}$, for instance. These are very different problems because the first one uses information about curves passing through very general points on the surface, while the second problem requires knowledge of specific curves passing through special points on the surface. See also Discussion 3.5.

For a sampling of the many results in this area, see [9, 3, 4, 15, 6, 10]. For a detailed account, see [5].

In this article, we address both the problems discussed above in the case of hyperelliptic surfaces. Our primary motivation is [10], where several results on Seshadri constants on hyperelliptic surfaces are proved.

Hyperelliptic surfaces are minimal surfaces of Kodaira dimension 0 and irregularity 1. They are realized as finite group quotients of products of two elliptic curves. These surfaces have been classified and are known to belong to one of seven different types. They all have Picard rank 2 and the free group $\mathrm{Num}(X)$ of divisors modulo numerical equivalence is well-understood. See Section 2 for more details.

Let X be a hyperelliptic surface and L be an ample line bundle on X. We first consider the problem of computing $\varepsilon(X, L)$ in subsection 3.1. In our main result Theorem 3.3 in this subsection, we show that $\varepsilon(X,L) \in \mathbb{Q}$ provided X is not of type 6. This partially answers [19, Question 1.6], which asks if $\varepsilon(X,L)$ is always rational for any pair (X,L). Some affirmative answers to this question are known ([3, 4, 22, 20, 12]), but it is open in general. In other results in this subsection, we also explicitly compute $\varepsilon(X,L)$ in some cases.

In subsection 3.2, we study the Seshadri constant of L at a very general point x. One of our main results, Theorem 3.11, says that $\varepsilon(X,L,1) \geq (0.93)\sqrt{L^2}$, or $\varepsilon(X,L,1)$ is equal to one of two easily computable natural numbers. Let L be of numerical type (a,b). Then depending on how a and b are related to each other, we either explicitly compute $\varepsilon(X,L,1)$ or show that $\varepsilon(X,L,1) \geq (0.93)\sqrt{L^2}$. We have such a result for each of the seven types of hyperelliptic surfaces. There are several results in the literature giving lower bounds for $\varepsilon(X,L,1)$ when X is an arbitrary surface and L is any ample line bundle. In Remark 3.19, we compare our bound $(0.93)\sqrt{L^2}$ with some existing bounds and note that it is often better.

We work over $\mathbb{C}$, the field of complex numbers. A *surface* is a two-dimensional smooth complex projective variety. When the surface X is clear from the context, we denote Seshadri constants simply by $\varepsilon(L,x), \varepsilon(L)$, or $\varepsilon(L,1)$.

2. Preliminaries

Definition 2.1. *A hyperelliptic surface X is a minimal smooth surface with Kodaira dimension $\kappa(X) = 0$ satisfying $h^1(X,\mathcal{O}_X) = 1$ and $h^2(X,\mathcal{O}_X) = 0$.*

Hyperelliptic surfaces are also known as *bielliptic surfaces* (cf. [7, 18]). We recall below some key properties of hyperelliptic surfaces that we use repeatedly. More details can be found in [7, 18]. We follow the notation in [18, 10].

There is an alternate characterization of hyperelliptic surfaces. A smooth surface X is hyperelliptic if and only if $X \cong (A \times B)/G$, where A and B are elliptic curves and G is a finite group of translation of A acting on B in such a way that $B/G \cong \mathbb{P}^1$.

We have the following diagram:

$$\begin{array}{ccc} X \cong (A \times B)/G & \xrightarrow{\Phi} & A/G \\ \Psi \big\downarrow & & \\ B/G \cong \mathbb{P}^1 & & \end{array}$$

In the above diagram Φ and Ψ are natural projections. The fibres of Φ are all smooth and isomorphic to B. The fibres of Ψ are all multiples of smooth elliptic curves, and all but finitely many of them are smooth and isomorphic to A.

Hyperelliptic surfaces were classified more than hundred years ago by G. Bagnera and M. de Franchis by analyzing the group G and its action on B. They showed that every hyperelliptic surface is of one of the seven types listed in the table below; see [7, V1.20].

Every hyperelliptic surface has Picard rank 2. Serrano [18] has described a basis for the free group $\mathrm{Num}(X)$ of divisors modulo numerical equivalence for each of the seven types of hyperelliptic surfaces. For each type, Serrano also lists the multiplicities $m_1, \dots, m_s$ of the singular fibres of Ψ, where s is the number of singular fibres.

Theorem 2.2. [18, Theorem 1.4]. *Let $X \cong (A \times B)/G$ be a hyperelliptic surface. A basis for the group Num(X) of divisors modulo numerical equivalence and the multiplicities of the singular fibres of $\Psi : X \to B/G$ in each type are given in the following table.*

Type of X	G	$m_1, m_2, \dots, m_s$	*Basis of Num(X)*
1	$\mathbb{Z}_2$	$2,2,2,2$	$A/2,\ B$
2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$2,2,2,2$	$A/2,\ B/2$
3	$\mathbb{Z}_4$	$2,4,4$	$A/4,\ B$
4	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$2,4,4$	$A/4$,$B/2$
5	$\mathbb{Z}_3$	$3,3,3$	$A/3, B$
6	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$3,3,3$	$A/3, B/3$
7	$\mathbb{Z}_6$	$2,3,6$	$A/6, B$

Let X be a hyperelliptic surface. Let $\mu = lcm(m_1, m_2, \dots, m_s)$ and let $\gamma = |G|$. By Serrano's theorem, a basis of Num(X) is given by A/μ, $(\mu/\gamma)B$.

Notation: We say that L *is a line bundle of type* (a, b) on X if L is numerically equivalent to $a \cdot A/\mu + b \cdot (\mu/\gamma)B$. If L is of type (a, b), we write $L \equiv (a, b)$.

We note the following properties of line bundles on X.

(1) $A^2 = 0, B^2 = 0, A \cdot B = \gamma$.
(2) A divisor $b \cdot (\mu/\gamma)B \equiv (0, b)$ is effective if and only if $b(\mu/\gamma) \in \mathbb{N}$ ([1, Proposition 5.2]).
(3) A line bundle of type (a, b) is ample if and only if $a > 0$ and $b > 0$ ([18, Lemma 1.3]).
(4) If C is an irreducible and reduced curve on X and $x \in C$ is a point of multiplicity m, then $C^2 \geq m^2 - m$.

The inequality in (4) follows from the genus formula, and the facts that the canonical divisor is numerically trivial on a hyperelliptic surface and that there are no rational curves on a hyperelliptic surface.

We also use the following important lower bound on self-intersection of a curve C passing through a very general point. See [9, 24, 15, 2], for instance.

Theorem 2.3. *Let X be a hyperelliptic surface and let C be an irreducible and reduced curve on X. Suppose that C passes through a very general point $x \in X$ with multiplicity $m \geq 2$. Then $C^2 \geq m^2 - m + 2$.*

3. Seshadri constants

In this section we first consider $\varepsilon(L)$ and then prove our results on $\varepsilon(L, 1)$.

3.1. **Results about $\varepsilon(L)$.**

Theorem 3.1. *Let X be a hyperelliptic surface of odd type (i.e., of type 1, 3, 5, or 7). Let $L \equiv (a, b)$ be an ample line bundle on X. Then $\varepsilon(L) = \min\{a, b\}$.*

Proof. We first prove that $\varepsilon(L, x) \geq \min\{a, b\}$ for any $x \in X$. We then show that equality holds for a suitable x.

Note that since X is a hyperelliptic surface of odd type, $\mu = \gamma$. Hence B is given by $(0, 1)$ in $\mathrm{Num}(X)$. On the other hand, A is given by $(2, 0), (4, 0), (3, 0)$ and $(6, 0)$ in types 1, 3, 5 and 7, respectively.

Since the fibres of $\Phi : X \to A/G$ cover X, are smooth and are isomorphic to B, there is a smooth curve which is numerically equivalent to $(0, 1)$ that contains any given point x. Similarly, the fibres of $\Psi : X \to B/G$ cover X, but they are not all smooth. The smooth fibres of Ψ are isomorphic to A and singular fibres are multiples of smooth fibres. The number of singular fibres and their multiplicities are completely determined by the type of X. See the table in Theorem 2.2.

Now let $x \in X$ be an arbitrary point. Let C be a reduced and irreducible curve on X passing through x with multiplicity $m \geq 1$. We consider three possibilities for C. First, it is a fibre of Φ; second, it is a fibre of Ψ; and third, it is different from the fibres of Φ and Ψ.

If C is a fibre of Φ, then C is smooth and is isomorphic to B and is numerically equivalent to $(0, 1)$. In this case, $m = 1$. So the Seshadri ratio is $L \cdot C = a$.

If C is a fibre of Ψ, then C is not necessarily smooth. Numerically, C is given by $(\mu, 0)$. The multiplicity m is determined by the table in Theorem 2.2. For instance, if X has type 1, then $m = 1$, or 2. Or, if X has type 3, then $m = 1, 2$, or 4. In this case, the Seshadri ratio is $\frac{L \cdot C}{m} = \frac{\mu b}{m}$.

Now let C be different from the fibres of Ψ and Φ. Let C be represented by (α, β) in $\mathrm{Num}(X)$. We use Bezout's theorem to bound the values of α and β. Since x is a point of a smooth fibre $(0, 1)$, we have $C \cdot (0, 1) = \alpha \geq m$. On the other hand, the fibre of Ψ containing x may not be smooth. In this case, Bezout's theorem gives $C \cdot (\mu, 0) = \mu\beta \geq mn$, where n is the multiplicity of the fibre of Ψ containing x. Thus we have $\frac{L \cdot C}{m} = \frac{a\beta + b\alpha}{m} \geq b + \frac{an}{\mu}$.

Since $\mu \geq m$ and $n \geq 1$, we conclude that the Seshadri ratio $\frac{L \cdot C}{m} \geq \min(a, b, b + \frac{a}{\mu})$ for any reduced irreducible curve C passing through x. Hence $\varepsilon(L, x) \geq \min(a, b, b + \frac{a}{\mu}) \geq \min(a, b)$.

Now let x be a point on a singular fibre of Ψ with the maximum possible multiplicity. For instance, if X has type 7, x is any point on a fibre of Ψ of multiplicity 6. Then, in the notation above, $m = n = \mu$. So $\varepsilon(L, x) = \min(a, b, a + b) = \min(a, b)$. This completes the proof of the theorem. □

Remark 3.2. The result in Theorem 3.1 is proved for hyperelliptic surfaces of type 1 in [10, Theorem 3.4] and our proof essentially follows from the arguments given by Farnik. In fact, Farnik gives a precise value for $\varepsilon(L, x)$ for any x and any ample line bundle L on a hyperelliptic surface of type 1.

Our next result partially answers [19, Question 1.6] for hyperelliptic surfaces. This question asks if $\varepsilon(X, L)$ is rational for any surface X and any ample line bundle L on X. So far an affirmative answer to this question has been found in some cases.

The case of quartic surfaces $X \subset \mathbb{P}^3$ and $L = \mathcal{O}_X(1)$ is considered in [3, Theorem]. It is proved that $\varepsilon(X, L) = 1, 4/3$ or 2, depending on certain geometric properties of X. In [4, Theorem A.1], it is proved that $\varepsilon(X, L)$ is rational if X is an abelian surface and L is any ample line bundle on X. The same result is shown for Enriques surfaces in [22, Theorem 3.3]. Finally, [20, 12] study minimal ruled surfaces. Such surfaces are geometrically ruled over a smooth curve C and one attaches a certain invariant $e \in \mathbb{Z}$ to them. If $e \geq 0$, then [20, Theorem 3.27] and [12, Theorem 4.14] show that $\varepsilon(X, L) \in \mathbb{Q}$ for any ample line bundle L on X.

Theorem 3.3. *Let X be a hyperelliptic surface of type different from 6 and let L be an ample line bundle on X. Then $\varepsilon(L)$ is rational.*

Proof. Let X be a hyperelliptic surface of type different from 6 and let $L \equiv (a, b)$ be an ample line bundle on X. If X has odd type then the assertion follows from Theorem 3.1.

X is of type 2: If $2a = b$, then $L^2 = 2ab$ is a perfect square and $\varepsilon(L)$ is a rational number (for instance, see [19, Corollary 1.8]).

If $b < 2a$, let x be a point on a singular fibre of Ψ. This fibre is numerically equivalent to $(2, 0)$ and x is a point of multiplicity 2 on it. So $\varepsilon(L, x) \leq \frac{(a,b)\cdot(2,0)}{2} = b < \sqrt{2ab} = \sqrt{L^2}$. It follows by [19, Lemma 1.7] that $\varepsilon(L) \in \mathbb{Q}$. On the other hand, if $b > 2a$, then let $x \in X$ be any point. Then x belongs to a fibre of Φ. Note that all the fibres of Φ are smooth and are numerically equivalent to $(0, 2)$. So $\varepsilon(L, x) \leq \frac{(a,b)\cdot(0,2)}{1} = 2a < \sqrt{L^2}$. Again it follows that $\varepsilon(L) \in \mathbb{Q}$. Note that in fact $\varepsilon(L, 1) \leq 2a < \sqrt{L^2}$, if $b > 2a$.

X is of type 4: As in the above case, if $2a = b$, then $\varepsilon(L) \in \mathbb{Q}$.

If $b < 2a$, let x be a point on a fibre of Ψ of multiplicity 4. Then $\varepsilon(L, x) \leq b < \sqrt{L^2}$. On the other hand, let $b > 2a$ and let x be any point. Consider the fibre of Φ containing x. This fibre is smooth and numerically equivalent to $(0, 2)$. Again as before, $\varepsilon(L, x) \leq 2a < \sqrt{L^2}$. □

We have the following result for type 6 hyperelliptic surfaces.

Theorem 3.4. *Let X be a hyperelliptic surface of type 6 and let $L \equiv (a, b)$ be an ample line bundle on X such that b is not in the interval $(2a, 9a/2)$. Then $\varepsilon(L) \in \mathbb{Q}$.*

Proof. If $b = 2a$ or $b = 9a/2$, then $L^2 = 2ab$ is a square and $\varepsilon(L) \in \mathbb{Q}$. So we assume that either $b < 2a$ or $b > 9a/2$.

If $b < 2a$, choose a point x on a singular fibre of Ψ. Then the fibre is represented numerically by $(3, 0)$ and the multiplicity of the fibre at x is 3. So $\varepsilon(L, x) \leq \frac{(a,b)\cdot(3,0)}{3} = b < \sqrt{2ab}$. If $b > 9a/2$, then choose any point x and consider a fibre of Φ containing it. We have $\varepsilon(L, x) \leq \frac{(a,b)\cdot(0,3)}{1} = 3a < \sqrt{2ab}$. Thus $\varepsilon(L) \in \mathbb{Q}$. □

Discussion 3.5. Let X be any surface, and let L be ample on X. If $x \in X$, then an easy upper bound for $\varepsilon(L, x)$ is given by $\frac{L \cdot C}{\mathrm{mult}_x C}$, where C is a curve containing x, *provided* this ratio is smaller than $\sqrt{L^2}$.

Of course, there are no such curves if $\varepsilon(X, L) = \sqrt{L^2}$. We note below why there are no obvious examples of such curves if X is a hyperelliptic surface of type 6 and $L \equiv (a, b)$ with $b \in (2a, 9a/2)$.

According to Theorems 3.3 and 3.4, if X is a hyperelliptic surface of type different from 6, or if X has type 6, but $L \equiv (a, b)$ with $b \notin (2a, 9a/2)$, then $\frac{L \cdot C}{\mathrm{mult}_x C} < \sqrt{L^2}$, for a suitable x and a suitable fibre C of Ψ or Φ. This in turn allows us to conclude $\varepsilon(X, L) \in \mathbb{Q}$ in these cases. It is also clear from the proof of Theorem 3.4 that if X has type 6 and $L \equiv (a, b)$ with $b \in (2a, 9a/2)$, then $\frac{L \cdot C}{\mathrm{mult}_x C} \geq \sqrt{L^2}$, for *any* fibre C of Ψ or Φ.

In general, for a surface X and an ample line bundle L on X, in order to conclude that $\varepsilon(X, L) \in \mathbb{Q}$, we must establish the existence of a suitable pair $x \in C$ for which $\frac{L \cdot C}{\mathrm{mult}_x C} < \sqrt{L^2}$. If X is a hyperelliptic surface of type 6 and $L \equiv (a, b)$ with $b \in (2a, 9a/2)$, there are no obvious candidates for such a pair. One needs more specific information about singular curves on such a surface.

We do however give a lower bound for $\varepsilon(L, x)$ for any x in the following proposition.

Proposition 3.6. *Let X be a hyperelliptic surface of type 6 and let $L \equiv (a, b)$ be an ample line bundle with $b \in (2a, 9a/2)$. Then $\varepsilon(L, x) \geq (0.7)\sqrt{L^2}$ for all $x \in X$.*

Proof. If $\varepsilon(L, x) < (0.7)\sqrt{L^2}$ for some $x \in X$, then $\varepsilon(L, x) = \frac{L \cdot C}{\mathrm{mult}_x C}$ for an irreducible and reduced curve $C \equiv (\alpha, \beta)$ containing x. Let $m = \mathrm{mult}_x C$. If $m = 1$, then $L \cdot C < \sqrt{L^2}$. Then the Hodge Index Theorem gives $L^2.C^2 \leq (L \cdot C)^2 < L^2$. So $C^2 = 2\alpha\beta < 1$. Thus $\alpha = 0$ or $\beta = 0$. Then C is a fibre of Φ or Ψ. But this is not possible, as mentioned in Discussion 3.5.

So assume $m \geq 2$. We know $C^2 \geq m^2 - m$ (see Section 2). Applying the Hodge Index Theorem again, we get $m^2 - m < (0.7)^2 m^2$, which gives $(0.51)m^2 - m < 0$. But this is not possible for $m \geq 2$. $\square$

We use the idea in the above proof again in Theorem 3.9.

Proposition 3.7. *Let X be a hyperelliptic surface of even type. Let $L \equiv (a, b)$ be an ample line bundle on X satisfying the following:*

(1) *$b \leq a$ if X is of type 2;*
(2) *$2b \leq a$ if X is of type 4 or 6.*

Then $\varepsilon(L) = b$.

Proof. First let X be of type 2. If x is a point on a singular fibre of Ψ, then as in the proof of Theorem 3.3, $\varepsilon(L, x) \leq b$.

Now let $x \in X$ be an arbitrary point. Then x is in a fibre of Φ which is represented by $(0, 2)$. The Seshadri ratio for this fibre is $\frac{L \cdot (0,2)}{1} = 2a \geq b$. The Seshadri ratio for any fibre of Ψ containing x is at least b. On the other hand, let $C \equiv (\alpha, \beta)$ be an irreducible and reduced curve, different from the fibres of Ψ or Φ, passing through x with multiplicity $m \geq 1$. Then (as in the proof of Theorem 3.1) Bezout's theorem gives $2\alpha \geq m$ and $2\beta \geq m$. So $\frac{L\cdot C}{m} = \frac{a\beta + b\alpha}{m} \geq \frac{a+b}{2} \geq b$. In other words, $\varepsilon(X, L, x) = \inf \frac{L\cdot C}{\mathrm{mult}_x C} \geq b$.

Thus $\varepsilon(L, x) \geq b$ for all $x \in X$ and $\varepsilon(L, x) \leq b$ if x is on a singular fibre of Ψ. It follows that $\varepsilon(L) = b$.

Now let X be of type 4 or 6. As in the above case, if x is on a singular fibre of Ψ, then $\varepsilon(L, x) \leq b$ (when X is of type 4, we take the point x on a fibre of multiplicity 4).

Now let $x \in X$ be arbitrary and let $C \equiv (\alpha, \beta)$ be an irreducible and reduced curve, different from the fibres of Ψ or Φ, passing through x with multiplicity $m \geq 1$. Then we have $4\beta \geq m$ and $2\alpha \geq m$ when X is of type 4 and $3\beta \geq m$ and $3\alpha \geq m$ when X is of type 6. In either case, $\frac{L\cdot C}{m} = \frac{a\beta + b\alpha}{m} \geq b$. As above, we conclude that $\varepsilon(L) = b$. □

Proposition 3.8. *Let X be a hyperelliptic surface of even type. Let $L \equiv (a, b)$ be an ample line bundle on X. Then the following statements hold:*

(1) *Let X be of type 2. If $b \geq 3a$, then $\varepsilon(L, x) = 2a$ for all $x \in X$.*
(2) *Let X be of type 4. If $b \geq 7a/2$, then $\varepsilon(L, x) = 2a$ for all $x \in X$.*
(3) *Let X be of type 6. If $b \geq 8a$, then $\varepsilon(L, x) = 3a$ for all $x \in X$.*

Proof. The proof is similar to the proof of Proposition 3.7, so we will only give a brief sketch.

First let X be of type 2. Let $x \in X$ be any point. Since a fibre of Φ contains x, we have $\varepsilon(L, x) \leq \frac{L\cdot(0,2)}{1} = 2a$. If x is on a singular fibre of Ψ, then the corresponding Seshadri ratio is $\frac{L\cdot(2,0)}{2} = b \geq 2a$. If x is on a smooth fibre of Ψ, then the corresponding Seshadri ratio is $\frac{L\cdot(2,0)}{1} = 2b \geq 2a$.

Now let $C \equiv (\alpha, \beta)$ be an irreducible and reduced curve, different from the fibres of Ψ or Φ, passing through x with multiplicity $m \geq 1$. Bezout's theorem gives $2\alpha \geq m$ and $2\beta \geq m$. So $\frac{L\cdot C}{m} = \frac{a\beta + b\alpha}{m} \geq \frac{a+b}{2} \geq 2a$, by hypothesis. So we conclude that $\varepsilon(L, x) = 2a$ for all $x \in X$.

The proof for types 4 and 6 is similar. □

3.2. **Results about** $\varepsilon(L, 1)$.

Theorem 3.9. *Let X be a hyperelliptic surface and let L be an ample line bundle on X. Suppose that $C \equiv (\alpha, \beta)$ is an irreducible, reduced curve with $\alpha \neq 0$, $\beta \neq 0$ and which passes through a very general point with multiplicity $m \geq 1$. Then $\frac{L\cdot C}{m} \geq (0.93)\sqrt{L^2}$.*

Proof. First, let $m = 1$. If $L \cdot C < (0.93)\sqrt{L^2}$, then the Hodge Index Theorem gives $C^2 < (0.93)^2$. So $C^2 = 2\alpha\beta = 0$, which violates the hypothesis on C.

So assume $m \geq 2$. Then we have $C^2 \geq m^2 - m + 2$, by Theorem 2.3. Again, the Hodge Index Theorem gives $m^2 - m + 2 < (0.93)^2 m^2$. So m satisfies the quadratic relation $(0.13)m^2 - m + 2 < 0$. But it is easy to see that the quadratic expression $(0.13)m^2 - m + 2$ is always positive, since it grows as m goes to ∞ or $-\infty$ and its discriminant is $1 - 8(0.13) = -0.04 < 0$. □

Remark 3.10. In the above proof, we used the fact that the quadratic $(1-\delta^2)m^2 - m + 2$ is positive for all $m \geq 1$, where $\delta = 0.93$. In order to get a better bound in Theorem 3.9, we have to increase δ. But this forces the above quadratic to become negative for some m. For instance, if $\delta = 0.94$, then the quadratic $(1-0.94^2)m^2 - m + 2$ is negative for $m = 4, 5$. Similarly, for $\delta = 0.99$, the quadratic $(1-0.99^2)m^2 - m + 2$ is negative for $2 \leq m \leq 48$. As δ approaches 1, the set $\{m \mid (1-\delta^2)m^2 - m + 2 < 0\}$ keeps increasing.

As δ approaches 1, more precise information about $L \cdot C$ for curves passing through very general points will be required to prove the inequality $\frac{L\cdot C}{m} \geq \delta\sqrt{L^2}$. This may be possible to do for specific line bundles L.

As a corollary to Theorem 3.9, we obtain our main theorem about $\varepsilon(L,1)$ for ample line bundles on hyperelliptic surfaces.

Theorem 3.11. *Let X be a hyperelliptic surface and let L be an ample line bundle on X. If $\varepsilon(L,1) < (0.93)\sqrt{L^2}$, then $\varepsilon(L,1) = \min(L \cdot A, L \cdot B)$.*

Proof. If $\varepsilon(L,1) \geq (0.93)\sqrt{L^2}$, then there is nothing to prove. Otherwise, we have $\varepsilon(L,1) = \frac{L\cdot C}{m}$, where C is a reduced and irreducible curve passing through a very general point with multiplicity m. Let $C \equiv (\alpha, \beta)$. By Theorem 3.9, either $\alpha = 0$ or $\beta = 0$. In other words, C is a fibre of Φ or Ψ. Since x is a very general point, we may assume that it does not lie on any of the finitely many singular fibres of Ψ. Thus C is smooth and isomorphic to B or A. Hence $\varepsilon(L,1) = \min(L \cdot A, L \cdot B)$. □

We next consider different types of hyperelliptic surfaces and prove specific results about $\varepsilon(L,1)$.

Theorem 3.12. *Let X be a hyperelliptic surface of type 1. Let $L \equiv (a,b)$ be an ample line bundle on X. Then $\varepsilon(L,1) = \min(a, 2b)$.*

Proof. We repeat the proof that is already essentially given in [10, Theorem 3.4]. This proof illustrates the special property of type 1 hyperelliptic surfaces in the sense that the Seshadri constants are always computed by the fibres of Φ or Ψ.

Note that when X has type 1, a fibre B of Φ is given by $(0,1)$ and a smooth fibre A of Ψ is given by $(2,0)$. So $L \cdot A = 2b$ and $L \cdot B = a$. Since a very general point $x \in X$ always belongs to a fibre B and a smooth fibre A, we have $\varepsilon(L,1) = \varepsilon(L,x) \leq \min(a, 2b)$.

Now if $C \equiv (\alpha, \beta)$ is a curve with $\alpha\beta \neq 0$ and it passes through a very general point with multiplicity m, then Bezout's theorem gives $\alpha \geq m$ and $\beta \geq m/2$. Thus $\frac{L\cdot C}{m} = \frac{a\beta + b\alpha}{m} \geq \frac{a}{2} + b \geq \min(a, 2b)$. It follows that $\varepsilon(L,1) = \min(a, 2b)$. □

Theorem 3.13. *Let X be a hyperelliptic surface of type 2. Let $L \equiv (a,b)$ be an ample line bundle on X. Then the following statements hold:*

1. *If* $\frac{2\min(a,b)}{(0.93)^2} \leq \max(\mathrm{a,b})$, *then* $\varepsilon(L,1) = 2\min(a,b)$.
2. *If* $\frac{2\min(a,b)}{(0.93)^2} \geq \max(\mathrm{a,b})$, *then* $\varepsilon(L,1) \geq (0.93)\sqrt{L^2}$.

Proof. Note that when X has type 2, a fibre B of Φ is given by $(0,2)$ and a smooth fibre A of Ψ is given by $(2,0)$. So $L \cdot A = 2b$ and $L \cdot B = 2a$. Since a very general point $x \in X$ always belongs to a fibre B and a smooth fibre A, we have $\varepsilon(L,1) = \varepsilon(L,x) \leq \min(2a,2b)$. Also, by Theorem 3.11, $\varepsilon(L,1) \geq (0.93)\sqrt{L^2}$ or $\varepsilon(L,1) = \min(2a,2b)$. Note that $L^2 = 2ab$.

We have

$$\begin{aligned} \frac{2\min(a,b)}{(0.93)^2} &\leq \max(\mathrm{a,b}) \\ \Leftrightarrow 4(\min(a,b))^2 &\leq (0.93)^2(2ab) \\ \Leftrightarrow 2\min(a,b) &\leq (0.93)\sqrt{2ab} \\ \Rightarrow \varepsilon(L,1) &= \min(2a,2b). \end{aligned}$$

On the other hand,

$\frac{2\min(a,b)}{(0.93)^2} \geq \max(\mathrm{a,b}) \Leftrightarrow 2\min(\mathrm{a,b}) \geq (0.93)\sqrt{2\mathrm{ab}} \Rightarrow \varepsilon(L,1) \geq (0.93)\sqrt{2ab}$. □

Theorem 3.14. *Let X be a hyperelliptic surface of type 3. Let $L \equiv (a,b)$ be an ample line bundle on X. Then the following statements hold:*

1. *If $b \leq \frac{a(0.93)^2}{8}$, then $\varepsilon(L,1) = 4b$.*
2. *If $\frac{a(0.93)^2}{8} \leq b \leq \frac{a}{2(0.93)^2}$, then $\varepsilon(L,1) \geq (0.93)\sqrt{L^2}$.*
3. *If $b \geq \frac{a}{2(0.93)^2}$, then $\varepsilon(L,1) = a$.*

Proof. Note that when X has type 3, a fibre B of Φ is given by $(0,1)$ and a smooth fibre A of Ψ is given by $(4,0)$. So $L \cdot A = 4b$ and $L \cdot B = a$. Since a very general point $x \in X$ always belongs to a fibre B and a smooth fibre A, we have $\varepsilon(L,1) = \varepsilon(L,x) \leq \min(a,4b)$. Also, by Theorem 3.11, $\varepsilon(L,1) \geq (0.93)\sqrt{L^2}$ or $\varepsilon(L,1) = \min(a,4b)$.

If $b \leq \frac{a(0.93)^2}{8}$, then clearly $4b \leq a$. Further $b \leq \frac{a(0.93)^2}{8} \Leftrightarrow 4b \leq (0.93)\sqrt{2ab}$. So $\varepsilon(L,1) = 4b$.

If $b \geq \frac{a}{2(0.93)^2}$, then clearly $a \leq 4b$. Further $b \geq \frac{a}{2(0.93)^2} \Leftrightarrow a \leq (0.93)\sqrt{2ab}$. So $\varepsilon(L,1) = a$.

Finally, if $\frac{a(0.93)^2}{8} \leq b \leq \frac{a}{2(0.93)^2}$, then $a \geq (0.93)\sqrt{2ab}$ and $4b \geq (0.93)\sqrt{2ab}$. So $\varepsilon(L,1) \geq (0.93)\sqrt{2ab}$. □

Theorem 3.15. *Let X be a hyperelliptic surface of type 4. Let $L \equiv (a,b)$ be an ample line bundle on X. Then the following statements hold:*

1. *If $b \leq \frac{a(0.93)^2}{8}$, then $\varepsilon(L,1) = 4b$.*
2. *If $\frac{a(0.93)^2}{8} \leq b \leq \frac{2a}{(0.93)^2}$, then $\varepsilon(L,1) \geq (0.93)\sqrt{L^2}$.*
3. *If $b \geq \frac{2a}{(0.93)^2}$, then $\varepsilon(L,1) = 2a$.*

Proof. Note that when X has type 4, a fibre B of Φ is given by $(0,2)$ and a smooth fibre A of Ψ is given by $(4,0)$. So $L \cdot A = 4b$ and $L \cdot B = 2a$. Since a very general point $x \in X$

always belongs to a fibre B and a smooth fibre A, we have $\varepsilon(L,1) = \varepsilon(L,x) \leq \min(2a, 4b)$. Also, by Theorem 3.11, $\varepsilon(L,1) \geq (0.93)\sqrt{L^2}$ or $\varepsilon(L,1) = \min(2a,4b)$.

If $b \leq \frac{a(0.93)^2}{8}$, then clearly $4b \leq 2a$. Further $b \leq \frac{a(0.93)^2}{8} \Leftrightarrow 4b \leq (0.93)\sqrt{2ab}$. So $\varepsilon(L,1) = 4b$.

If $b \geq \frac{2a}{(0.93)^2}$, then clearly $2a \leq 4b$. Further $b \geq \frac{2a}{(0.93)^2} \Leftrightarrow 2a \leq (0.93)\sqrt{2ab}$. So $\varepsilon(L,1) = 2a$.

Finally, if $\frac{a(0.93)^2}{8} \leq b \leq \frac{2a}{(0.93)^2}$, then $2a \geq (0.93)\sqrt{2ab}$ and $4b \geq (0.93)\sqrt{2ab}$. So $\varepsilon(L,1) \geq (0.93)\sqrt{2ab}$. □

Theorem 3.16. *Let X be a hyperelliptic surface of type 5. Let $L \equiv (a,b)$ be an ample line bundle on X. Then the following statements hold:*

(1) *If $b \leq \frac{2a(0.93)^2}{9}$, then $\varepsilon(L,1) = 3b$.*
(2) *If $\frac{2a(0.93)^2}{9} \leq b \leq \frac{a}{2(0.93)^2}$, then $\varepsilon(L,1) \geq (0.93)\sqrt{L^2}$.*
(3) *If $b \geq \frac{a}{2(0.93)^2}$, then $\varepsilon(L,1) = a$.*

Proof. Note that when X has type 5, a fibre B of Φ is given by $(0,1)$ and a smooth fibre A of Ψ is given by $(3,0)$. So $L \cdot A = 3b$ and $L \cdot B = a$. Since a very general point $x \in X$ always belongs to a fibre B and a smooth fibre A, we have $\varepsilon(L,1) = \varepsilon(L,x) \leq \min(a,3b)$. Also, by Theorem 3.11, $\varepsilon(L,1) \geq (0.93)\sqrt{L^2}$ or $\varepsilon(L,1) = \min(a,3b)$.

If $b \leq \frac{2a(0.93)^2}{9}$, then clearly $3b \leq a$. Further $b \leq \frac{2a(0.93)^2}{9} \Leftrightarrow 3b \leq (0.93)\sqrt{2ab}$. So $\varepsilon(L,1) = 3b$.

If $b \geq \frac{a}{2(0.93)^2}$, then clearly $a \leq 3b$. Further $b \geq \frac{a}{2(0.93)^2} \Leftrightarrow a \leq (0.93)\sqrt{2ab}$. So $\varepsilon(L,1) = a$.

Finally, if $\frac{2a(0.93)^2}{9} \leq b \leq \frac{a}{2(0.93)^2}$, then $a \geq (0.93)\sqrt{2ab}$ and $3b \geq (0.93)\sqrt{2ab}$. So $\varepsilon(L,1) \geq (0.93)\sqrt{2ab}$. □

Theorem 3.17. *Let X be a hyperelliptic surface of type 6. Let $L \equiv (a,b)$ be an ample line bundle on X. Then the following statements hold:*

(1) *If $\frac{9\min(a,b)}{2(0.93)^2} \leq \max(\text{a,b})$, then $\varepsilon(L,1) = 3\min(a,b)$.*
(2) *If $\frac{9\min(a,b)}{2(0.93)^2} \geq \max(a,b)$, then $\varepsilon(L,1) \geq (0.93)\sqrt{L^2}$.*

Proof. Note that when X has type 6, a fibre B of Φ is given by $(0,3)$ and a smooth fibre A of Ψ is given by $(3,0)$. So $L \cdot A = 3b$ and $L \cdot B = 3a$. Since a very general point $x \in X$ always belongs to a fibre B and a smooth fibre A, we have $\varepsilon(L,1) = \varepsilon(L,x) \leq \min(3a,3b)$. Also, by Theorem 3.11, $\varepsilon(L,1) \geq (0.93)\sqrt{L^2}$ or $\varepsilon(L,1) = \min(3a,3b)$.

We have

$$\begin{aligned} \frac{9\min(a,b)}{2(0.93)^2} &\leq \text{max(a, b)} \\ \Leftrightarrow 9(\min(a,b))^2 &\leq (0.93)^2(2ab) \\ \Leftrightarrow 3\min(a,b) &\leq (0.93)\sqrt{2ab} \\ \Rightarrow \varepsilon(L,1) &= \min(3a,3b). \end{aligned}$$

On the other hand,

$\frac{9\min(a,b)}{2(0.93)^2} \geq \max(a,b) \Leftrightarrow 3\min(a,b) \geq (0.93)\sqrt{2ab} \Rightarrow \varepsilon(L,1) \geq (0.93)\sqrt{2ab}$. □

Theorem 3.18. *Let X be a hyperelliptic surface of type 7. Let $L \equiv (a,b)$ be an ample line bundle on X. Then the following statements hold:*

(1) *If $b \leq \frac{a(0.93)^2}{18}$, then $\varepsilon(L,1) = 6b$.*
(2) *If $\frac{a(0.93)^2}{18} \leq b \leq \frac{a}{2(0.93)^2}$, then $\varepsilon(L,1) \geq (0.93)\sqrt{L^2}$.*
(3) *If $b \geq \frac{a}{2(0.93)^2}$, then $\varepsilon(L,1) = a$.*

Proof. Note that when X has type 7, a fibre B of Φ is given by $(0,1)$ and a smooth fibre A of Ψ is given by $(6,0)$. So $L \cdot A = 6b$ and $L \cdot B = a$. Since a very general point $x \in X$ always belongs to a fibre B and a smooth fibre A, we have $\varepsilon(L,1) = \varepsilon(L,x) \leq \min(a,6b)$. Also, by Theorem 3.11, $\varepsilon(L,1) \geq (0.93)\sqrt{L^2}$ or $\varepsilon(L,1) = \min(a,6b)$.

If $b \leq \frac{a(0.93)^2}{18}$, then clearly $6b \leq a$. Further $b \leq \frac{a(0.93)^2}{18} \Leftrightarrow 6b \leq (0.93)\sqrt{2ab}$. So $\varepsilon(L,1) = 6b$.

If $b \geq \frac{a}{2(0.93)^2}$, then clearly $a \leq 6b$. Further $b \geq \frac{a}{2(0.93)^2} \Leftrightarrow a \leq (0.93)\sqrt{2ab}$. So $\varepsilon(L,1) = a$.

Finally, if $\frac{a(0.93)^2}{18} \leq b \leq \frac{a}{2(0.93)^2}$, then $a \geq (0.93)\sqrt{2ab}$ and $6b \geq (0.93)\sqrt{2ab}$. So $\varepsilon(L,1) \geq (0.93)\sqrt{2ab}$. □

Remark 3.19. We compare the result in Theorem 3.11 with some bounds in the literature. There has been a lot of interest in finding good lower bound for $\varepsilon(L,1)$. See, for instance, [16, 14, 23, 21, 11].

Let X be any surface and let L be an ample line bundle on X. It is known that $\varepsilon(X,L,1) \geq \sqrt{\frac{7}{9}}\sqrt{L^2}$, or X is fibred by Seshadri curves, or X is a cubic surface in $\mathbb{P}^3$ and $L = \mathcal{O}_X(1)$; see [21, Corollary 3.3]. Since $\sqrt{\frac{7}{9}}$ is approximately 0.88, the bound we give in Theorem 3.11 is better.

Another recent result in this direction is contained in [11]. Let $d := L^2$ and suppose that d is not a square. Then the equation $y^2 - dx^2 = 1$ is known as *Pell's equation*. If $x = p, y = q$ is a solution to this equation, then [11, Theorem 1.3] shows that $\varepsilon(L,1) \geq \frac{p}{q}d$ or $\varepsilon(X,L,1)$ is contained in a finite set $\text{Exc}(d;p,q)$ of rational numbers which are easy to list. Though this bound is often better than $(0.93)\sqrt{L^2}$, the set $\text{Exc}(d;p,q)$ is typically large.

As an example, let X be a hyperelliptic surface of type 6 and let $L \equiv (5,11)$. Then $d = L^2 = 110$ and $\sqrt{L^2} \sim 10.49$. By Theorem 3.17, $\varepsilon(X,L,1) \geq (0.93)\sqrt{110} \sim 9.75$. On the other hand, $(2,21)$ is a solution to Pell's equation $y^2 - 110x^2 = 1$. So by [11, Theorem 1.3], $\varepsilon(X,L,1) \geq \frac{220}{21} \sim 10.48$, or $\varepsilon(X,L,1) \in \mathrm{Exc}(110;2,21)$. Though 10.48 is a much better approximation to $\sqrt{L^2}$ compared to our 9.75, the exceptional set $\mathrm{Exc}(110;2,21)$ is large and it is not easy in general to lower the number of possibilities. In this case, $\mathrm{Exc}(110;2,21) = \{1,2,\ldots,10\} \cup \{\frac{r}{s} \mid 1 \leq \frac{r}{s} < \frac{220}{21} \text{ and } 2 \leq s < 21^2 = 441\}$.

We also note that our results give precise values of $\varepsilon(X,L,1)$ in many cases. For example, if X is hyperelliptic of type 6 and $L \equiv (5,b)$ and $b \geq 27$, then $\varepsilon(X,L,1) = 15$, by Theorem 3.17.

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References

[1] Aprodu, Marian, *An Appell-Humbert theorem for hyperelliptic surfaces*, J. Math. Kyoto Univ. 38 (1998), no. 1, 101-121.

[2] Bastianelli, Francesco, *Remarks on the nef cone on symmetric products of curves*, Manuscripta Math. 130 (2009), no. 1, 113-120.

[3] Bauer, Thomas, *Seshadri constants of quartic surfaces*, Math. Ann. 309 (1997), no. 3, 475-481.

[4] Bauer, Thomas, *Seshadri constants and periods of polarized abelian varieties*, With an appendix by the author and Tomasz Szemberg, Math. Ann. 312 (1998), no. 4, 607-623.

[5] Bauer, Thomas; Di Rocco, Sandra; Harbourne, Brian; Kapustka, Michał; Knutsen, Andreas; Syzdek, Wioletta; Szemberg, Tomasz, *A primer on Seshadri constants*, Interactions of classical and numerical algebraic geometry, 33-70, Contemp. Math., 496, Amer. Math. Soc., Providence, RI, 2009.

[6] Bauer, Thomas; Szemberg, Tomasz, *Seshadri constants on surfaces of general type*, Manuscripta Math. 126 (2008), no. 2, 167-175.

[7] Beauville, Arnaud, *Complex algebraic surfaces*, Translated from the 1978 French original by R. Barlow, with assistance from N. I. Shepherd-Barron and M. Reid. Second edition. London Mathematical Society Student Texts, 34. Cambridge University Press, Cambridge, 1996.

[8] Demailly, Jean-Pierre, *Singular Hermitian metrics on positive line bundles*, Complex algebraic varieties (Bayreuth, 1990), 87-104, Lecture Notes in Math., 1507, Springer, Berlin, 1992.

[9] Ein, Lawrence; Lazarsfeld, Robert, *Seshadri constants on smooth surfaces*, Journées de Géométrie Algébrique d'Orsay (Orsay, 1992), Astérisque No. 218 (1993), 177-186.

[10] Farnik, Łucja, *A note on Seshadri constants of line bundles on hyperelliptic surfaces*, Arch. Math. (Basel) 107 (2016), no. 3, 227-237.

[11] Farnik, Łucja; Szemberg, Tomasz; Szpond, Justyna; Tutaj-Gasińska, Halszka, *Restrictions on Seshadri constants on surfaces*, Taiwanese J. Math. 21 (2017), no. 1, 27-41.

[12] García, Luis Fuentes, *Seshadri constants on ruled surfaces: the rational and the elliptic cases*, Manuscripta Math. 119 (2006), no. 4, 483-505.

[13] Hartshorne, Robin, *Ample subvarieties of algebraic varieties*, Notes written in collaboration with C. Musili. Lecture Notes in Mathematics, Vol. 156 Springer-Verlag, Berlin-New York 1970.

[14] Hwang, Jun-Muk; Keum, JongHae, *Seshadri-exceptional foliations*, Math. Ann. 325 (2003), no. 2, 287-297.

[15] Knutsen, Andreas Leopold; Syzdek, Wioletta; Szemberg, Tomasz, *Moving curves and Seshadri constants*, Math. Res. Lett. 16 (2009), no. 4, 711-719.

[16] Nakamaye, Michael, *Seshadri constants and the geometry of surfaces*, J. Reine Angew. Math. 564 (2003), 205-214.

[17] Oguiso, Keiji, *Seshadri constants in a family of surfaces*, Math. Ann. 323 (2002), no. 4, 625-631.

[18] Serrano, Fernando, *Divisors of bielliptic surfaces and embeddings in* $\mathbb{P}^4$, Math. Z. 203 (1990), no. 3, 527-533.
[19] Strycharz-Szemberg, Beata; Szemberg, Tomasz, *Remarks on the Nagata conjecture*, Serdica Math. J. 30 (2004), no. 2-3, 405-430.
[20] Syzdek, Wioletta, *Seshadri constants and geometry of surfaces*, Ph.D thesis, 2005, University of Duisburg-Essen.
[21] Syzdek, Wioletta; Szemberg, Tomasz, *Seshadri fibrations of algebraic surfaces*, Math. Nachr. 283 (2010), no. 6, 902-908.
[22] Szemberg, Tomasz, *On positivity of line bundles on Enriques surfaces*, Trans. Amer. Math. Soc. 353 (2001), no. 12, 4963-4972.
[23] Szemberg, Tomasz; Tutaj-Gasińska, Halszka, *Seshadri fibrations on algebraic surfaces*, Ann. Acad. Paedagog. Crac. Stud. Math. 4 (2004), 225-229.
[24] Xu, Geng, *Curves in* $\mathbb{P}^2$ *and symplectic packings*, Math. Ann. 299 (1994), 609-613.

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