# Lecture 21, 5 November 2024

Madhavan Mukund

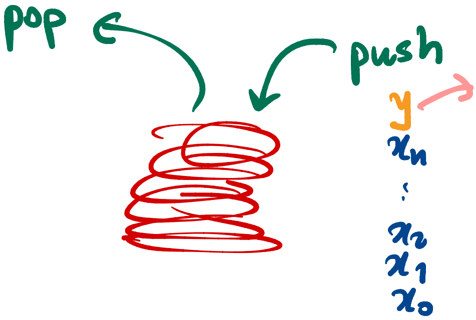
https://www.cmi.ac.in/~madhavan

Programming and Data Structures with Python

Lecture 21, 05 Nov 2023

# Stack

- Stack is a last-in, first-out sequence
- push(s,x) — add x to stack s
- pop(s) — return most recently added element



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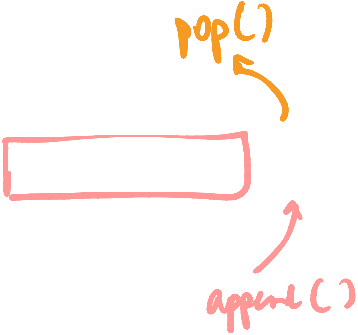
  - `push(s,x)` is `s.append(x)`
  - `s.pop()` — Python built-in, returns last element



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- Stack defined using classes: `s.push(x)`, `s.pop()`

Calling functions

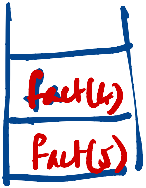
fact(n)

:

n * fact(n-1)

fact(5) ⇝ 5 * fact(4)

⇝ fact(3)



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Stack is a list
with restricted access

$$\underbrace{(s.push(x))}_{s'}.pop() == x$$

ABSTRACT data type

# Stack

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Need

s.empty() → True
False

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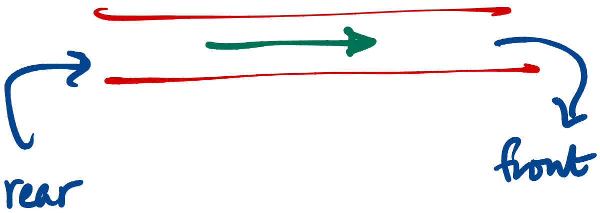
- Maintain stack as list, push and pop from the right
  - `push(s,x)` is `s.append(x)`
  - `s.pop()` — Python built-in, returns last element
- Stack defined using classes: `s.push(x)`, `s.pop()`
- Stacks are natural to keep track of local variables through function calls
  - Each function call pushes current frame onto a stack
  - When function exits, pop its frame off the stack

# Queue

- First-in, first-out sequence
- `addq(q,x)` — adds `x` to rear of queue `q`
- `removeq(q)` — removes element at head of `q`

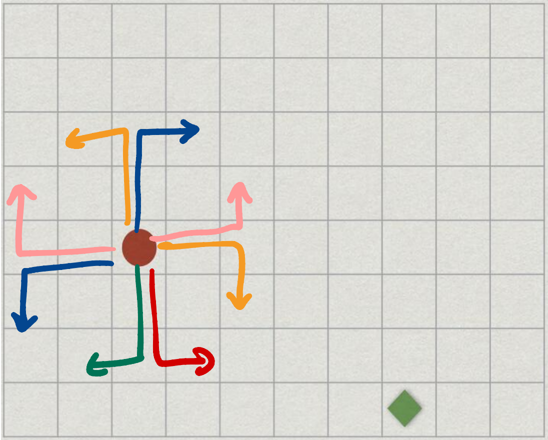


# Queue

- First-in, first-out sequence
- `addq(q,x)` — adds `x` to rear of queue `q`
- `removeq(q)` — removes element at head of `q`
- Using Python lists, left is rear, right is front
  - `addq(q,x)` is `q.insert(0,x)`
    - `insert(j,x)`, insert `x` before position `j`
  - `removeq(q)` is `q.pop()`

# Systematic exploration

- Rectangular $m \times n$ grid
- Chess knight starts at $(sx, sy)$ ●
- Usual knight moves
- Can it reach a target square $(tx, ty)$? ♦



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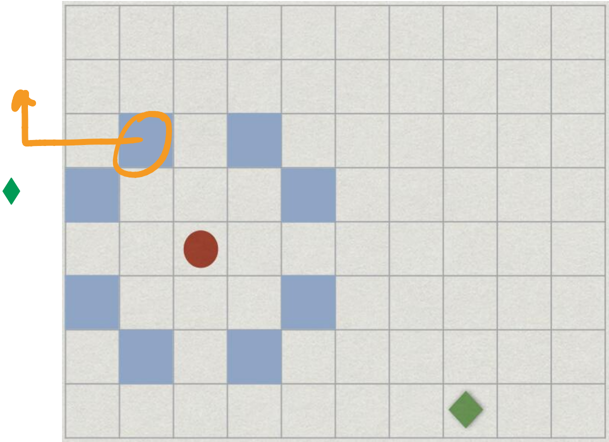

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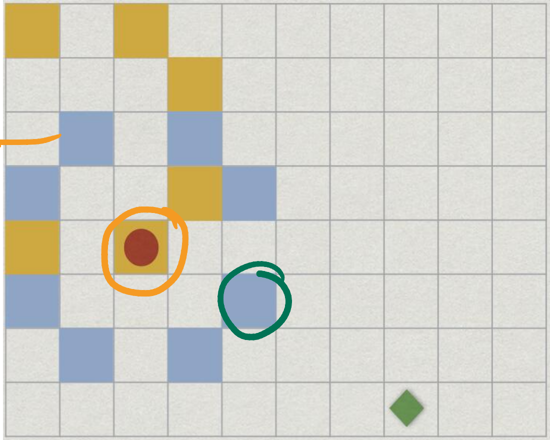

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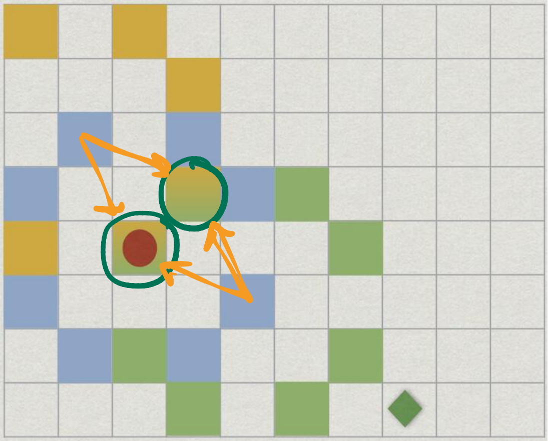

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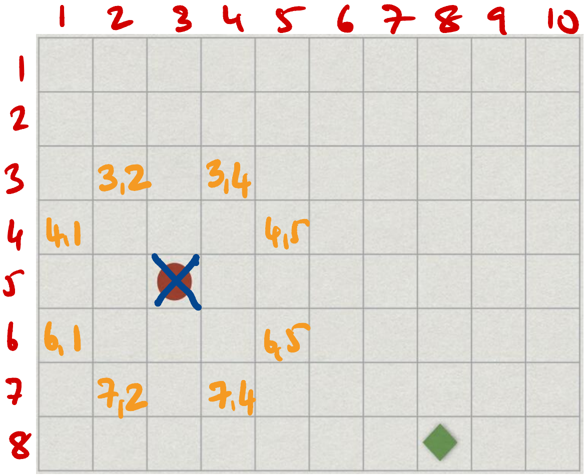
[(5,3)]

Mark it
Add unmarked
nbrs to queue

(7,2),(7,4),(6,5),(4,5),(3,4),(3,2)]



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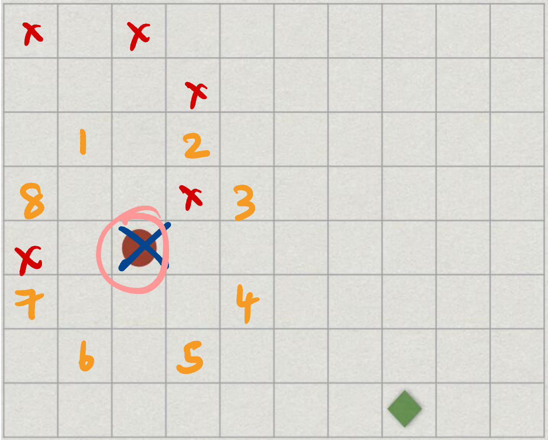
[8 ..4, 3, 2, 1]



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- $X1$ — all squares reachable in one move from $(sx, sy)$
- $X2$ —- all squares reachable from $X1$ in one move

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- When the queue is empty, we have finished

# Dealing with priorities

## Job scheduler

- A job scheduler maintains a list of pending jobs with their priorities

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- How should the scheduler maintain the list of pending jobs and their priorities?

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- When the processor is free, the scheduler picks out the job with maximum priority in the list and schedules it
- New jobs may join the list at any time
- How should the scheduler maintain the list of pending jobs and their priorities?

## Priority queue

- Need to maintain a collection of items with priorities to optimise the following operations
- `delete_max()`
  - Identify and remove item with highest priority
  - Need not be unique
- `insert()`
  - Add a new item to the collection

# Implementing priority queues with one dimensional structures

Unordered list

Insert? = append()

$O(1)$

Delete max?

- Scan list

Comparison

$O(n)$

- `delete_max()`
  - Identify and remove item with highest priority
  - Need not be unique
- `insert()`
  - Add a new item to the list

# Implementing priority queues with one dimensional structures

- Unsorted list
  - `insert()` is $O(1)$
  - `delete_max()` is $O(n)$

Sorted (Think insertion sort)

Insert $O(n)$

DeleteMax = pop() = $O(1)$

- `delete_max()`
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  - `insert()` is $O(n)$

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# Implementing priority queues with one dimensional structures

- Unsorted list
  - `insert()` is $O(1)$
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- Sorted list
  - `delete_max()` is $O(1)$
  - `insert()` is $O(n)$
- Processing $n$ items requires $O(n^2)$

- `delete_max()`
  - Identify and remove item with highest priority
  - Need not be unique
- `insert()`
  - Add a new item to the list

# Moving to two dimensions

## First attempt

- Assume $N$ processes enter/leave the queue

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- Assume $N$ processes enter/leave the queue
- Maintain a $\sqrt{N} \times \sqrt{N}$ array

$N = 25$

| | | | | |
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| 3 | 19 | 23 | 35 | 58 |
| 12 | 17 | 25 | 43 | 67 |
| 10 | 13 | 20 | | |
| 11 | 16 | 28 | 49 | |
| 6 | 14 | | | |

→

# Moving to two dimensions

## First attempt

- Assume $N$ processes enter/leave the queue
- Maintain a $\sqrt{N} \times \sqrt{N}$ array
- Each row is in sorted order

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# insert()

- Keep track of the size of each row

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# insert()

- Keep track of the size of each row
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  - Use size of row to determine

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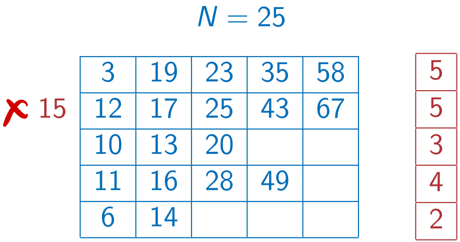

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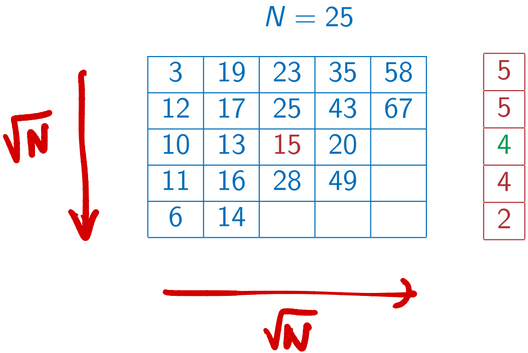

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# insert()

- Keep track of the size of each row
- Insert into the first row that has space
  - Use size of row to determine
- Insert 15
- Takes time $O(\sqrt{N})$
  - Scan size column to locate row to insert, $O(\sqrt{N})$
  - Insert into the first row with free space, $O(\sqrt{N})$

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# delete_max()

- Maximum in each row is the last element

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# delete_max()

- Maximum in each row is the last element
- Position is available through size column

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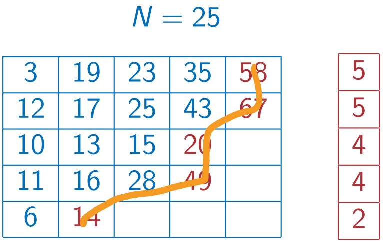

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# delete_max()

- Maximum in each row is the last element
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- Identify the maximum amongst these

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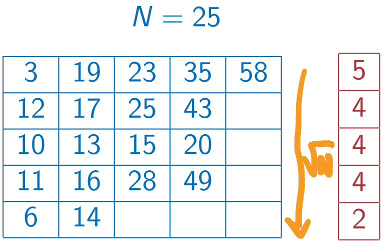
# delete_max()

- Maximum in each row is the last element
- Position is available through size column
- Identify the maximum amongst these
- Delete it
- Again $O(\sqrt{N})$
  - Find the maximum among last entries, $O(\sqrt{N})$
  - Delete it, $O(1)$



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# Summary

- 2D $\sqrt{N} \times \sqrt{N}$ array with sorted rows
  - `insert()` is $O(\sqrt{N})$
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  - Processing $N$ items is $O(N\sqrt{N})$

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- Can we do better?
- Maintain a special binary tree — heap
  - Height $O(\log N)$
  - `insert()` is $O(\log N)$
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  - Processing $N$ items is $O(N \log N)$

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  - Height $O(\log N)$
  - `insert()` is $O(\log N)$
  - `delete_max()` is $O(\log N)$
  - Processing $N$ items is $O(N \log N)$
- Flexible — need not fix $N$ in advance

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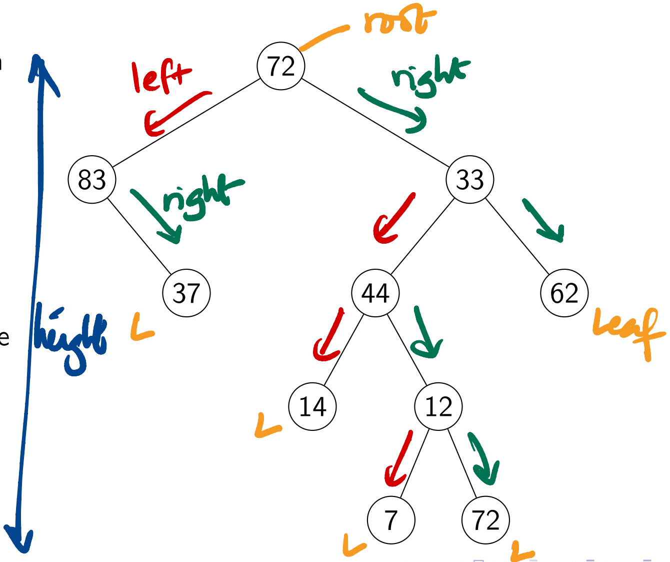
# Binary trees

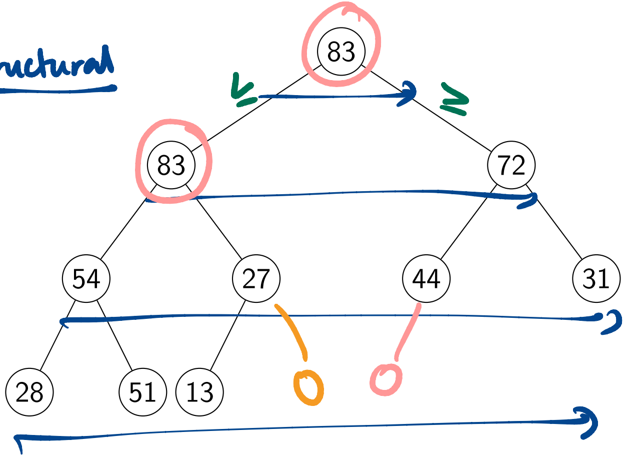
- Values are stored as nodes in a rooted tree
- Each node has up to two children
    - Left child and right child
    - Order is important
- Other than the root, each node has a unique parent
- Leaf node — no children
- Size — number of nodes
- Height — number of levels



# Heap

- Binary tree filled level by level, left to right
- The value at each node is at least as big the values of its children
  - max-heap

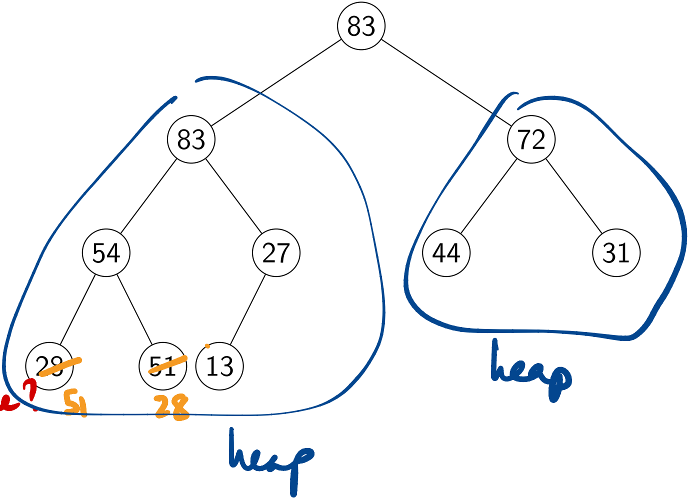
Structural



# Heap

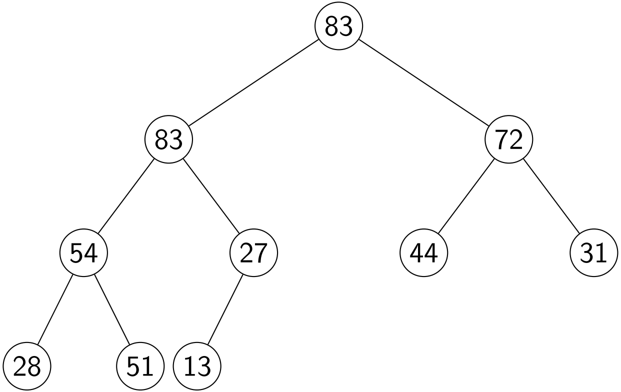
- Binary tree filled level by level, left to right
- The value at each node is at least as big the values of its children
  - max-heap
- Binary tree on the right is an example of a heap

Where is maximum value?



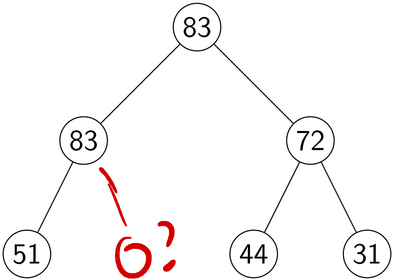
# Heap

- Binary tree filled level by level, left to right
- The value at each node is at least as big the values of its children
  - max-heap
- Binary tree on the right is an example of a heap
- Root always has the largest value
  - By induction, because of the max-heap property



# Non-examples

No "holes" allowed



# Non-examples

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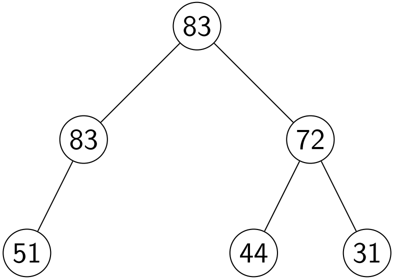
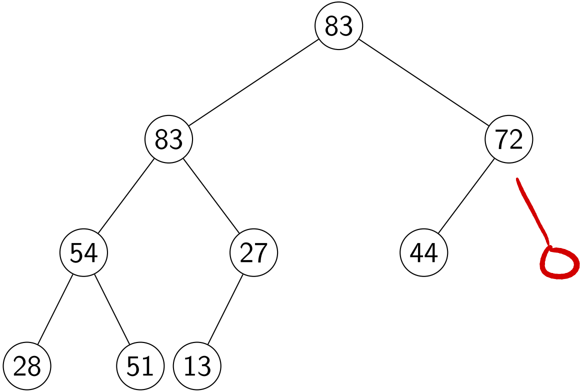
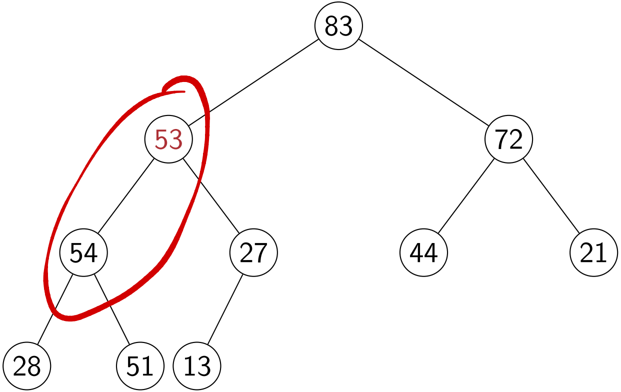


Cannot leave a level incomplete



# Non-examples

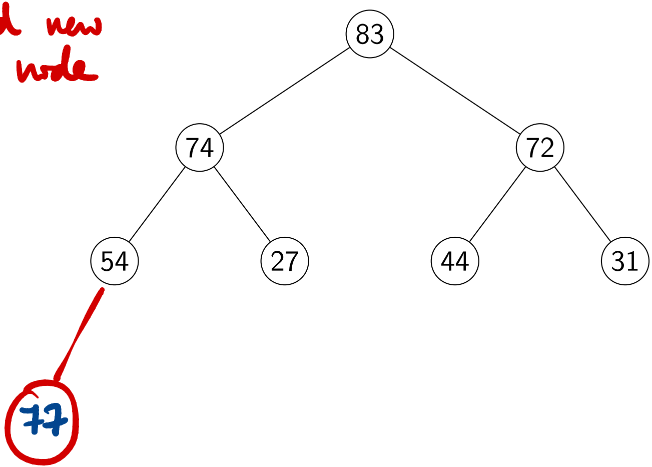
Heap property is violated



# insert()

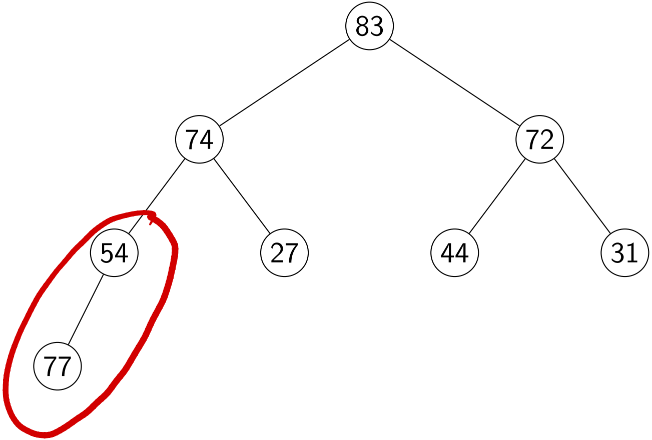
- insert(77)

Add new node



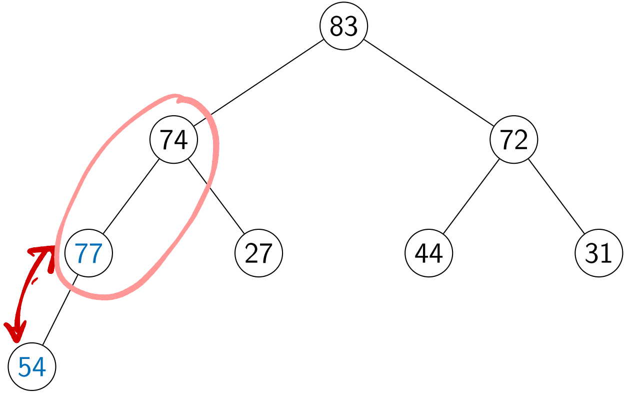
# insert()

- `insert(77)`
- Add a new node at dictated by heap structure



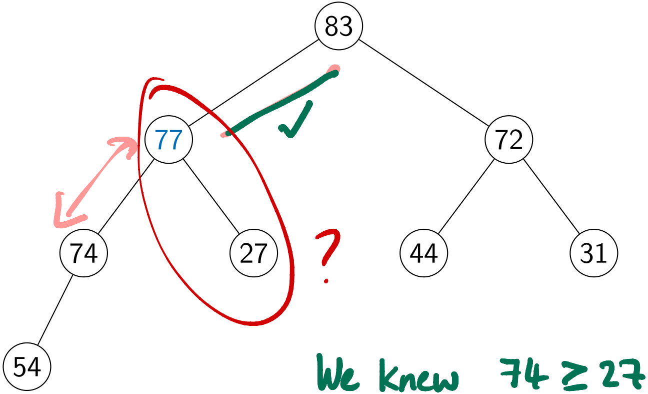
# insert()

- insert(77)
- Add a new node at dictated by heap structure
- Restore the heap property along path to the root



# insert()

- insert(77)
- Add a new node at dictated by heap structure
- Restore the heap property along path to the root



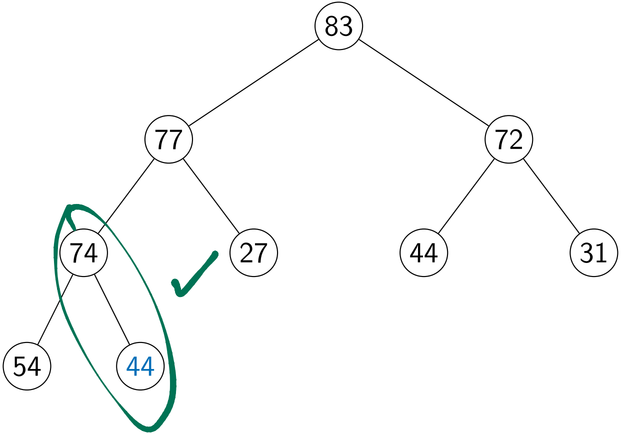
We knew 74 ≥ 27

And 77 ≥ 74

So 77 ≥ 27

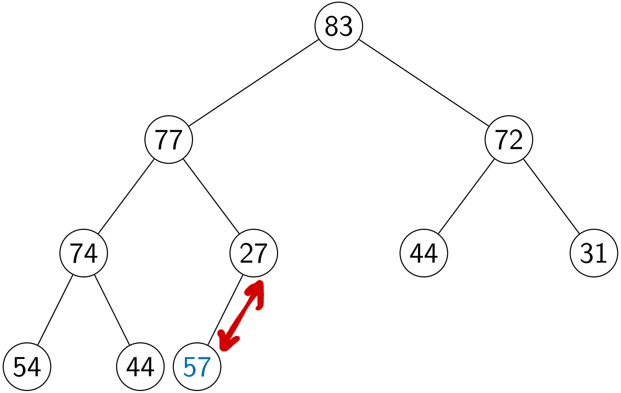
# insert()

- insert(77)
- Add a new node at dictated by heap structure
- Restore the heap property along path to the root
- insert(44)



# insert()

- insert(77)
- Add a new node at dictated by heap structure
- Restore the heap property along path to the root
- insert(44)
- insert(57)



# insert()

- insert(77)
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- insert(44)
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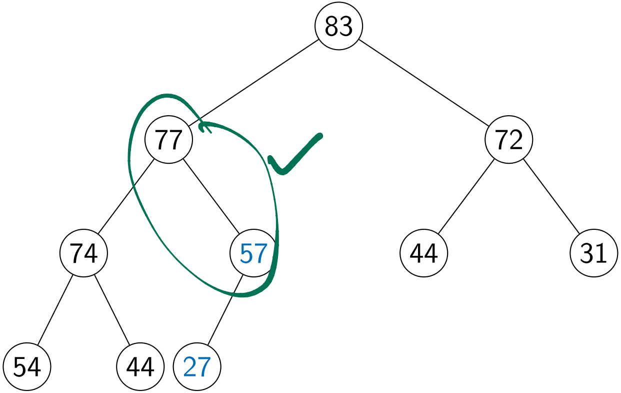
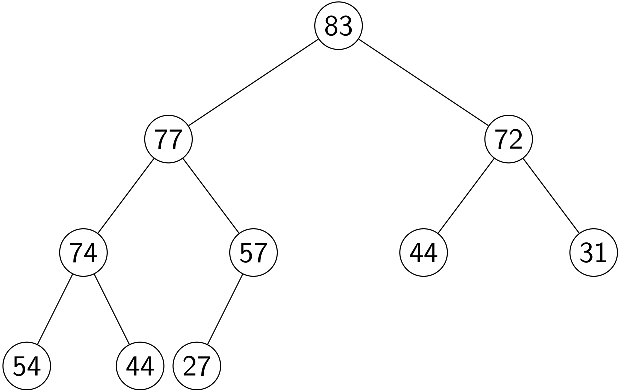


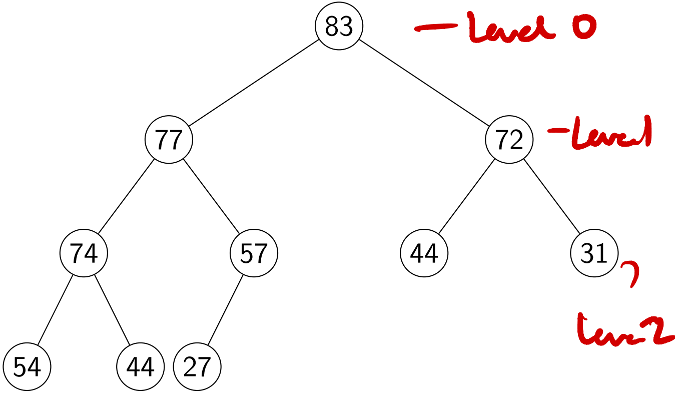
# Complexity of `insert()`

- Need to walk up from the leaf to the root
  - Height of the tree



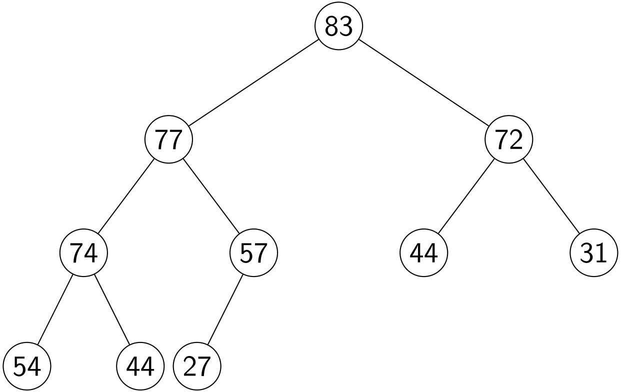
# Complexity of insert()

- Need to walk up from the leaf to the root
  - Height of the tree
- Number of nodes at level 0 is $2^0 = 1$



# Complexity of `insert()`

- Need to walk up from the leaf to the root
  - Height of the tree
- Number of nodes at level $0$ is $2^0 = 1$
- Number of nodes at level $j$ is $2^j$



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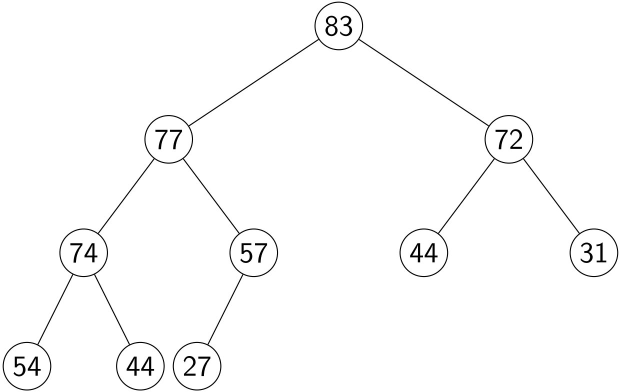
- Need to walk up from the leaf to the root
  - Height of the tree
- Number of nodes at level $0$ is $2^0 = 1$
- Number of nodes at level $j$ is $2^j$
- If we fill $k$ levels, $2^0 + 2^1 + \cdots + 2^{k-1} = 2^k - 1$ nodes

k height

$2^k - 1$ size



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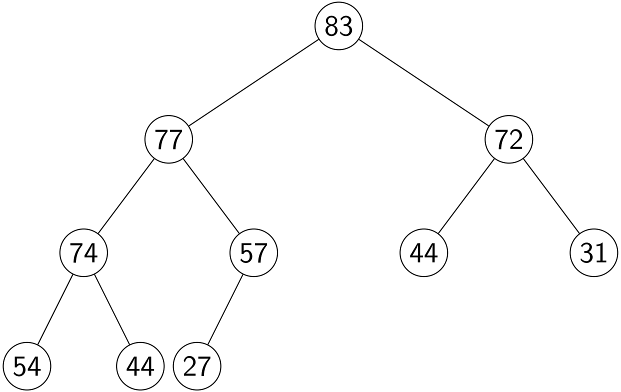
- If we have $N$ nodes, at most $1 + \log N$ levels



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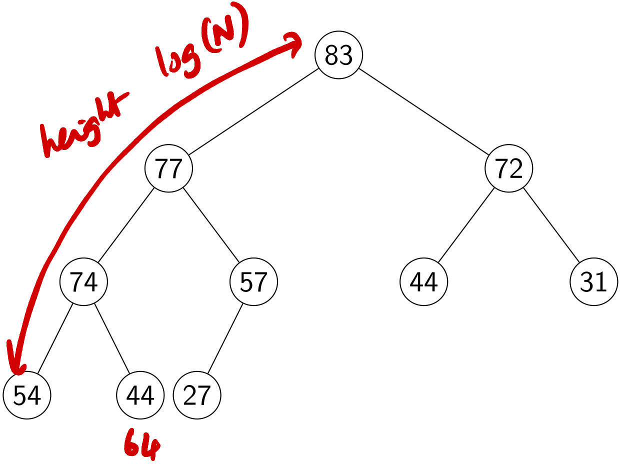
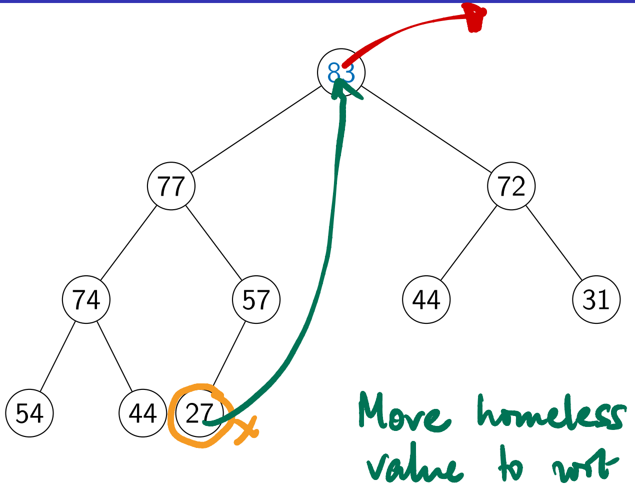
- If we fill $k$ levels, $2^0 + 2^1 + \cdots + 2^{k-1} = 2^k - 1$ nodes
- If we have $N$ nodes, at most $1 + \log N$ levels
- `insert()` is $O(\log N)$



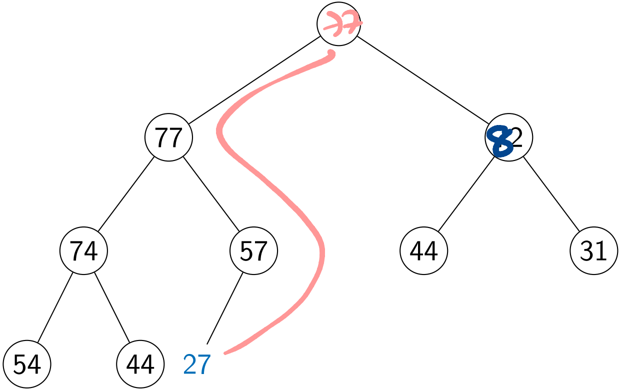
# delete_max()

- Maximum value is always at the root



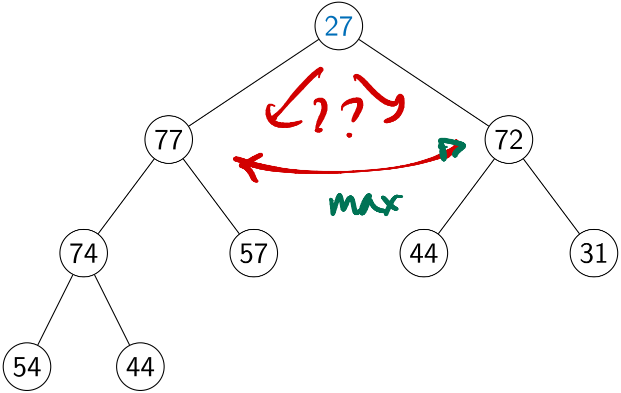
# delete_max()

- Maximum value is always at the root
- After we delete one value, tree shrinks
  - Node to delete is rightmost at lowest level



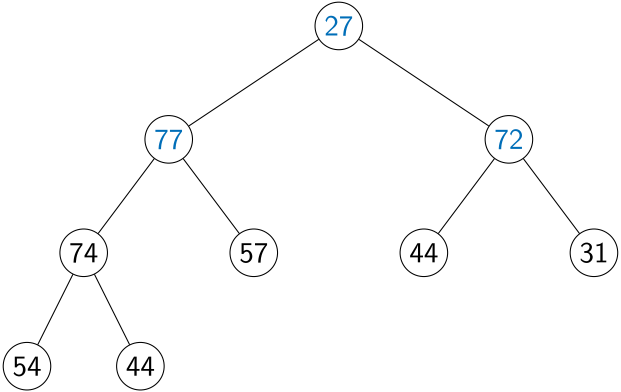
# delete_max()

- Maximum value is always at the root
- After we delete one value, tree shrinks
  - Node to delete is rightmost at lowest level
- Move "homeless" value to the root



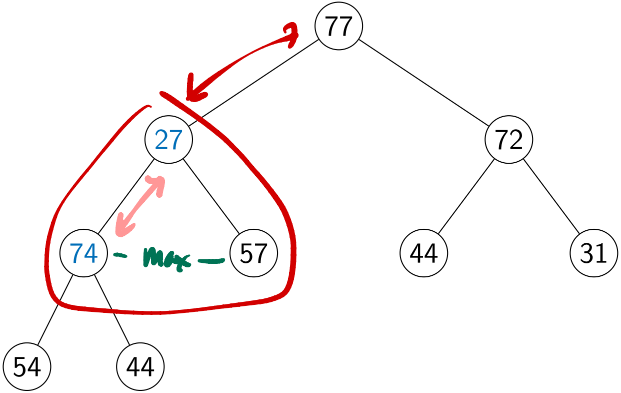
# delete_max()

- Maximum value is always at the root
- After we delete one value, tree shrinks
  - Node to delete is rightmost at lowest level
- Move "homeless" value to the root
- Restore the heap property downwards



# delete_max()

- Maximum value is always at the root
- After we delete one value, tree shrinks
  - Node to delete is rightmost at lowest level
- Move "homeless" value to the root
- Restore the heap property downwards
- Only need to follow a single path down
  - Again $O(\log N)$



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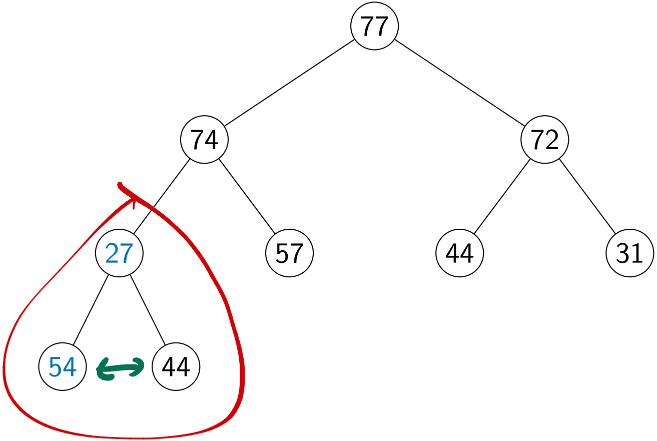

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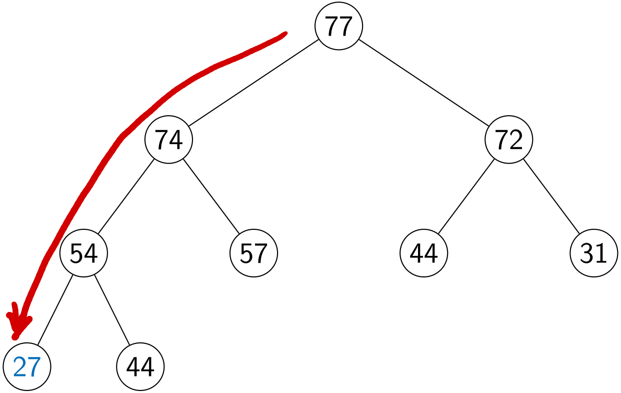
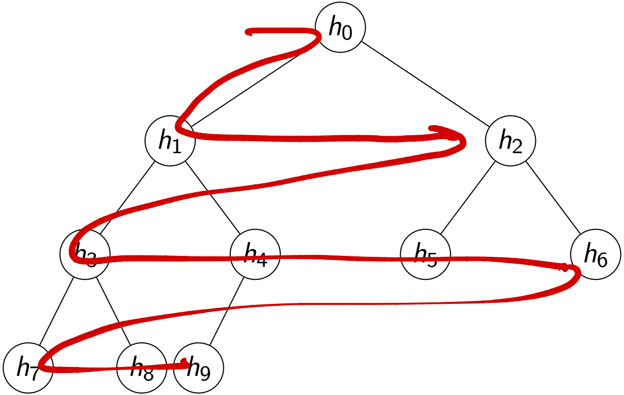


# Implementation

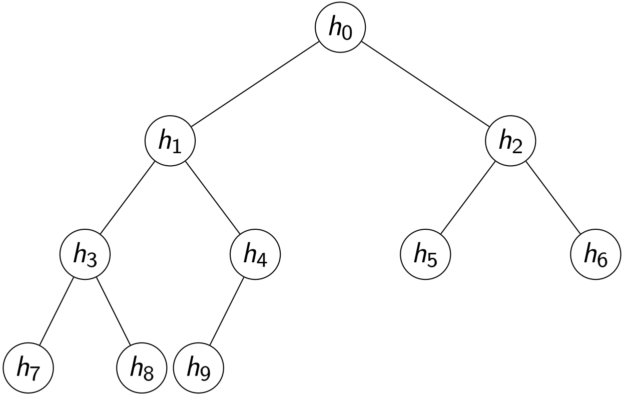
- Number the nodes top to bottom left right
- Store as a list
  `H = [h0,h1,h2,...,h9]`
- Children of `H[i]` are at `H[2*i+1]`, `H[2*i+2]`
- Parent of `H[i]` is at `H[(i-1)//2]`, for `i > 0`



leaf? 2i+1, 2i+2 are beyond N

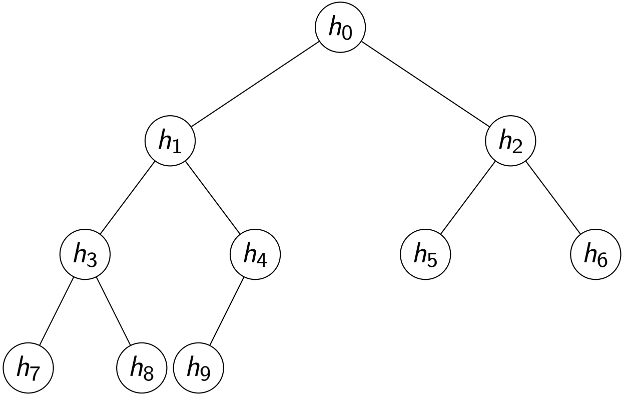
# Building a heap — `heapify()`

- Convert a list `[v0,v1,...,vN]` into a heap



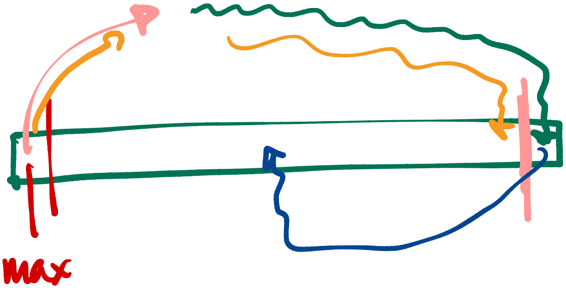
# Building a heap — `heapify()`

- Convert a list `[v0,v1,...,vN]` into a heap
- Simple strategy
  - Start with an empty heap
  - Repeatedly apply `insert(vj)`
  - Total time is $O(N \log N)$



# Heap sort

- Start with an unordered list



delete m

# Heap sort

- Start with an unordered list
- Build a heap — ~~$O(n)$~~ $O(n \log n)$

# Heap sort

- Start with an unordered list
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- Start with an unordered list
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- After each `delete_max()`, heap shrinks by $1$

# Heap sort

- Start with an unordered list
- Build a heap — $O(n)$
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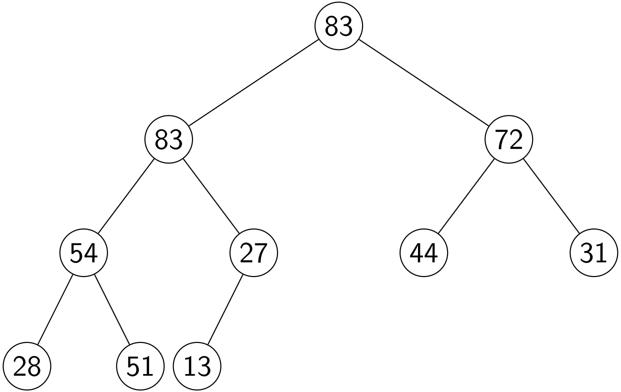
- Store maximum value at the end of current heap

# Heap sort

- Start with an unordered list
- Build a heap — $O(n)$
- Call `delete_max()` $n$ times to extract elements in descending order — $O(n \log n)$
- After each `delete_max()`, heap shrinks by $1$
- Store maximum value at the end of current heap
- In place $O(n \log n)$ sort

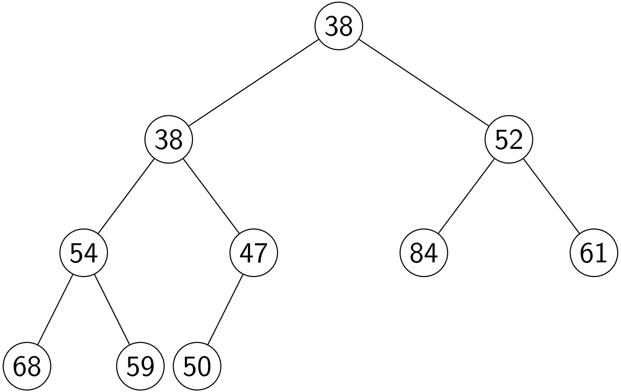
# Summary

- Heaps are a tree implementation of priority queues
  - `insert()` is $O(\log N)$
  - `delete_max()` is $O(\log N)$
  - `heapify()` builds a heap in $O(N)$



# Summary

- Heaps are a tree implementation of priority queues
  - `insert()` is $O(\log N)$
  - `delete_max()` is $O(\log N)$
  - `heapify()` builds a heap in $O(N)$
- Can invert the heap condition
  - Each node is smaller than its children
  - min-heap
  - `delete_min()` rather than `delete_max()`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