Lecture 24, 21 November 2024

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Programming and Data Structures with Python

Lecture 24, 21 Nov 2024

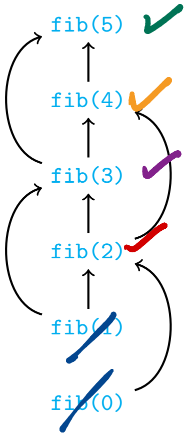
Dynamic programming

- Anticipate the structure of subproblems
  - Derive from inductive definition
  - Dependencies are acyclic
- Solve subproblems in appropriate order
  - Start with base cases — no dependencies
  - Evaluate a value after all its dependencies are available
  - Fill table iteratively
  - Never need to make a recursive call

Evaluating `fib(5)`

$fib(n) = fib(n-1) + fib(n-2)$

i	$fib(i)$
0	0
1	1
2	1
3	2
4	3
5	5



Longest common subsequence

- Subsequence — can drop some letters in between
- Given two strings, find the (length of the) longest common subsequence
  - `"secret"`, `"secretary"` — `"secret"`, length 6
  - `"bisect"`, `"trisect"` — `"isect"`, length 5
  - `"bisect"`, `"secret"` — `"sect"`, length 4
  - `"director"`, `"secretary"` — `"ectr"`, `"retr"`, length 4

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  - "bisect", "secret" — "sect", length 4
  - "director", "secretary" — "ectr", "retr", length 4
- LCS is the longest path connecting non-zero LCW entries, moving right/down

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	0	0	0	0	0	0	0
1	i	0	0	0	0	0	0	0
2	s	3	0	0	0	0	0	0
3	e	0	2	0	0	1	0	0
4	c	0	0	1	0	0	0	0
5	t	0	0	0	0	0	1	0
6	•	0	0	0	0	0	0	0

Applications

- Analyzing genes
  - DNA is a long string over `A`, `T`, `G`, `C`
  - Two species are similar if their DNA has long common subsequences
- `diff` command in Unix/Linux
  - Compares text files
  - Find the longest matching subsequence of lines
  - Each line of text is a "character"

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	i	**0**	**0**	**0**	**0**	**0**	**0**	**0**
2	s	**3**	**0**	**0**	**0**	**0**	**0**	**0**
3	e	**0**	**2**	**0**	**0**	**1**	**0**	**0**
4	c	**0**	**0**	**1**	**0**	**0**	**0**	**0**
5	t	**0**	**0**	**0**	**0**	**0**	**1**	**0**
6	•	**0**	**0**	**0**	**0**	**0**	**0**	**0**

Inductive structure

- $u = a_0 a_1 \ldots a_{m-1}$, $v = b_0 b_1 \ldots b_{n-1}$
- $LCS(i,j)$ — length of longest common subsequence in $a_i a_{i+1} \ldots a_{m-1}$, $b_j b_{j+1} \ldots b_{n-1}$

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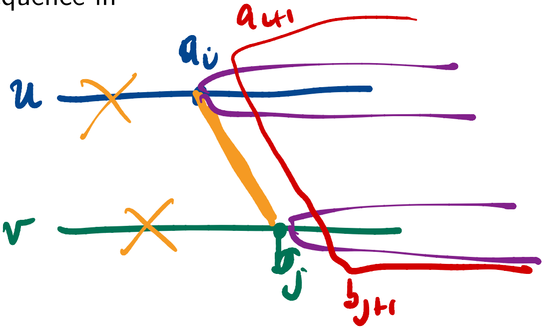
  - $LCS(i,j) = 1 + LCS(i{+}1, j{+}1)$



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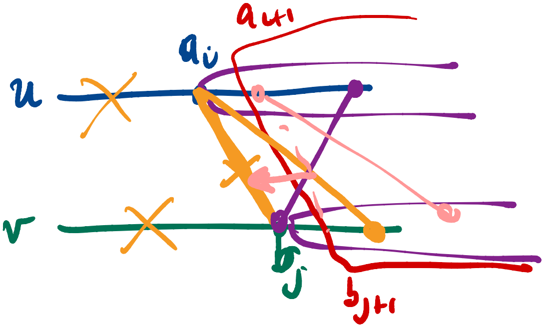

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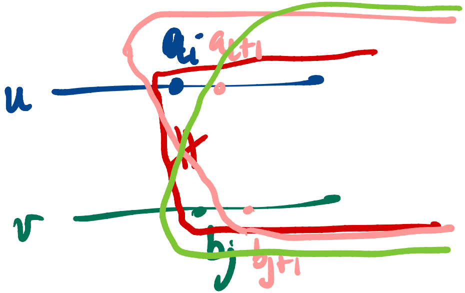
  - Which one should we drop?



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  - Which one should we drop?
  - $LCS(i,j) = \max(LCS(i, j{+}1), LCS(i{+}1, j))$
- Base cases as with LCW
  - $LCS(i,n) = 0$ for all $0 \leq i \leq m$
  - $LCS(m,j) = 0$ for all $0 \leq j \leq n$

Subproblem dependency

- Subproblems are $LCS(i, j)$, for $0 \leq i \leq m$, $0 \leq j \leq n$

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							
1	i							
2	s							
3	e							
4	c							
5	t							
6	•							

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							**0**
1	i							**0**
2	s							**0**
3	e							**0**
4	c							**0**
5	t							**0**
6	•							**0**

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b						**1**	**0**
1	i						**1**	**0**
2	s						**1**	**0**
3	e						$\neq$ **1**	**0**
4	c						$\neq$ **1**	**0**
5	t						= **1**	**0**
6	•						**0**	**0**

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0	b					2	1	0
1	i					2	1	0
2	s					2	1	0
3	e					2	1	0
4	c					1	1	0
5	t					1	1	0
6	•					0	0	0

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		s	e	c	r	e	t	•
0	b				**2**	**2**	**1**	**0**
1	i				**2**	**2**	**1**	**0**
2	s				**2**	**2**	**1**	**0**
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		s	e	c	r	e	t	•
0	b			**2**	**2**	**2**	**1**	**0**
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3	e			**2**	**2**	**2**	**1**	**0**
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2	s	4	3	2	2	2	1	0
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Reading off the solution

- Trace back the path by which each entry was filled

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	3	2	2	2	1	0
1	i	4	3	2	2	2	1	0
2	s	4	3	2	2	2	1	0
3	e	3	3	2	2	2	1	0
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Reading off the solution

- Trace back the path by which each entry was filled
- Each diagonal step is an element of LCS

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	3	2	2	2	1	0
1	i	4	3	2	2	2	1	0
2	s	4	3	2	2	2	1	0
3	e	3	3	2	2	2	1	0
4	c	2	2	2	1	1	1	0
5	t	1	1	1	1	1	1	0
6	•	0	0	0	0	0	0	0

Implementation

```
def LCS(u,v):
  import numpy as np
  (m,n) = (len(u),len(v))
  lcs = np.zeros((m+1,n+1))

  for j in range(n-1,-1,-1):
    for i in range(m-1,-1,-1):
      if u[i] == v[j]:
        lcs[i,j] = 1 + lcs[i+1,j+1]
      else:
        lcs[i,j] = max(lcs[i+1,j],
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  return(lcs[0,0])
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Complexity

- Again $O(mn)$, using dynamic programming or memoization
  - Fill a table of size $O(mn)$
  - Each table entry takes constant time to compute

Document similarity

- "The students were able to appreciate the concept optimal substructure property and its use in designing algorithms"
- "The lecture taught the students to appreciate how the concept of optimal substructures can be used in designing algorithms"

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  - Insert a character
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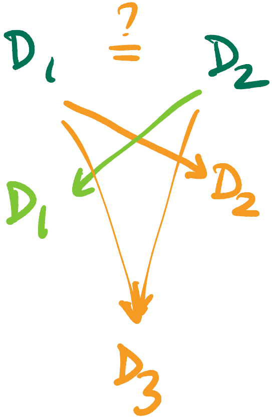

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Edit distance

- Minimum number of edit operations needed
- In our example, 24 characters inserted, 18 ~~deleted~~, 2 ~~substituted~~
- Edit distance is at most 44

Edit distance

- Minimum number of editing operations needed to transform one document to the other
  - Insert a character
  - Delete a character
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- Applications
  - Suggestions for spelling correction
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  - Delete b, i in bisect and r, e in secret

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- Deleting a letter from u is equivalent to inserting it in v
  - `bisect`, `secret` — LCS is `sect`
  - Delete `b`, `i` in `bisect` and `r`, `e` in `secret`
  - Delete `b`, `i` and then insert `r`, `e` in `bisect`
- From LCS, we can compute edit distance without substitution

Inductive structure for edit distance

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- Recall LCS

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- Edit distance — aim is to transform u to v

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ED(i,j)

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  $LCS(i+1, j)\ ]$

- Edit distance — aim is to transform u to v
- If $a_i = b_j$, nothing to be done

Inductive structure for edit distance

- $u = a_0 a_1 \ldots a_{m-1}$, $v = b_0 b_1 \ldots b_{n-1}$

- Recall LCS
- $LCS(i,j)$ — length of longest common subsequence in $a_i a_{i+1} \ldots a_{m-1}$, $b_j b_{j+1} \ldots b_{n-1}$
- If $a_i = b_j$, $LCS(i,j) = 1 + LCS(i{+}1, j{+}1)$
- If $a_i \neq b_j$, $LCS(i,j) = \max[\ LCS(i, j{+}1),\ LCS(i{+}1, j)\ ]$

- Edit distance — aim is to transform u to v
- If $a_i = b_j$, nothing to be done $i{+}1, j{+}1$
- If $a_i \neq b_j$, best among

Inductive structure for edit distance

- $u = a_0 a_1 \ldots a_{m-1}$, $v = b_0 b_1 \ldots b_{n-1}$

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  - Substitute a_i by b_j

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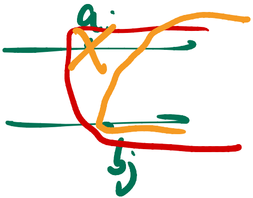

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  - Insert b_j before a_i

Inductive structure for edit distance

- $u = a_0 a_1 \ldots a_{m-1}$, $v = b_0 b_1 \ldots b_{n-1}$
- $ED(i,j)$ — edit distance for $a_i a_{i+1} \ldots a_{m-1}$, $b_j b_{j+1} \ldots b_{n-1}$
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- If $a_i \neq b_j$,
  $ED(i,j) = 1 + \min[\ ED(i+1,j+1),\ ED(i+1,j),\ ED(i,j+1)\ ]$

substitute

delete a_i

insert b_j before a_i

Inductive structure for edit distance

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  $ED(i,j) = ED(i{+}1, j{+}1)$
- If $a_i \neq b_j$,
  $$ED(i,j) = 1 + \min[\; ED(i{+}1,j{+}1),\; ED(i{+}1,j),\; ED(i,j{+}1) \;]$$
- Base cases
  - $ED(m,n) = 0$
  - $ED(i,n) = m - i$ for all $0 \leq i \leq m$
    Delete $a_i a_{i+1} \dots a_{m-1}$ from u
  - $ED(m,j) = n - j$ for all $0 \leq j \leq n$
    Insert $b_j b_{j+1} \dots b_{n-1}$ into u

Subproblem dependency

- Subproblems are $ED(i, j)$, for $0 \leq i \leq m$, $0 \leq j \leq n$

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- Table of $(m+1) \cdot (n+1)$ values

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							
1	i							
2	s							
3	e							
4	c							
5	t							
6	•							

Subproblem dependency

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b							
1	i							
2	s							
3	e							
4	c							
5	t							
6	•							

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	
0	b							**6**
1	i							**5**
2	s							**4**
3	e							**3**
4	c							**2**
5	t							**1**
6		6	5	4	3	2	1	**0**

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b						**5**	**6**
1	i						**4**	**5**
2	s					1+	**3**	**4**
3	e					min	**2**	**3**
4	c						**1**	**2**
5	t						**0**	**1**
6	•						**1**	**0**

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b					4	5	6
1	i					3	4	5
2	s					2	3	4
3	e					1	2	3
4	c					1	1	2
5	t					1	0	1
6	•					2	1	0

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b				**4**	**4**	**5**	**6**
1	i				**3**	**3**	**4**	**5**
2	s				**2**	**2**	**3**	**4**
3	e				**2**	**1**	**2**	**3**
4	c				**2**	**1**	**1**	**2**
5	t				**2**	**1**	**0**	**1**
6	•				**3**	**2**	**1**	**0**

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b			**4**	**4**	**4**	**5**	**6**
1	i			**3**	**3**	**3**	**4**	**5**
2	s			**3**	**2**	**2**	**3**	**4**
3	e			**3**	**2**	**1**	**2**	**3**
4	c			**2**	**2**	**1**	**1**	**2**
5	t			**3**	**2**	**1**	**0**	**1**
6	•			**4**	**3**	**2**	**1**	**0**

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b		4	4	4	4	5	6
1	i		4	3	3	3	4	5
2	s		3	3	2	2	3	4
3	e		2	3	2	1	2	3
4	c		3	2	2	1	1	2
5	t		4	3	2	1	0	1
6	•		5	4	3	2	1	0

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		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	**4**	**4**	**4**	**4**	**4**	**5**	**6**
1	i	**3**	**4**	**3**	**3**	**3**	**4**	**5**
2	s	**2**	**3**	**3**	**2**	**2**	**3**	**4**
3	e	**3**	**2**	**3**	**2**	**1**	**2**	**3**
4	c	**4**	**3**	**2**	**2**	**1**	**1**	**2**
5	t	**5**	**4**	**3**	**2**	**1**	**0**	**1**
6	•	**6**	**5**	**4**	**3**	**2**	**1**	**0**

Subproblem dependency

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Reading off the solution

- Transform `bisect` to `secret`

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	4	4	4	4	5	6
1	i	3	4	3	3	3	4	5
2	s	2	3	3	2	2	3	4
3	e	3	2	3	2	1	2	3
4	c	4	3	2	2	1	1	2
5	t	5	4	3	2	1	0	1
6	•	6	5	4	3	2	1	0

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Reading off the solution

- Transform `bisect` to `secret`
- Delete `b`

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	4	4	4	4	5	6
1	i	3	4	3	3	3	4	5
2	s	2	3	3	2	2	3	4
3	e	3	2	3	2	1	2	3
4	c	4	3	2	2	1	1	2
5	t	5	4	3	2	1	0	1
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Reading off the solution

- Transform `bisect` to `secret`
- Delete `b` , Delete `i`

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	4	4	4	4	5	6
1	i	3	4	3	3	3	4	5
2	s	2	3	3	2	2	3	4
3	e	3	2	3	2	1	2	3
4	c	4	3	2	2	1	1	2
5	t	5	4	3	2	1	0	1
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Reading off the solution

- Transform `bisect` to `secret`
- Delete `b` , Delete `i` , Insert `r`

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	4	4	4	4	5	6
1	i	3	4	3	3	3	4	5
2	s	2	3	3	2	2	3	4
3	e	3	2	3	2	1	2	3
4	c	4	3	2	2	1	1	2
5	t	5	4	3	2	1	0	1
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Reading off the solution

- Transform `bisect` to `secret`
- Delete `b` , Delete `i` , Insert `r` , Insert `e`

		0	1	2	3	4	5	6
		s	e	c	r	e	t	•
0	b	4	4	4	4	4	5	6
1	i	3	4	3	3	3	4	5
2	s	2	3	3	2	2	3	4
3	e	3	2	3	2	1	2	3
4	c	4	3	2	2	1	1	2
5	t	5	4	3	2	1	0	1
6	•	6	5	4	3	2	1	0

Implementation

```
def ED(u,v):
  import numpy as np
  (m,n) = (len(u),len(v))
  ed = np.zeros((m+1,n+1))

  for i in range(m-1,-1,-1):
    ed[i,n] = m-i
  for j in range(n-1,-1,-1):
    ed[m,j] = n-j

  for j in range(n-1,-1,-1):
    for i in range(m-1,-1,-1):
      if u[i] == v[j]:
        ed[i,j] = ed[i+1,j+1]
      else:
        ed[i,j] = 1 + min(ed[i+1,j+1],
                          ed[i,j+1],
                          ed[i+1,j])
  return(ed[0,0])
```

Initialization

Implementation

```
def ED(u,v):
  import numpy as np
  (m,n) = (len(u),len(v))
  ed = np.zeros((m+1,n+1))

  for i in range(m-1,-1,-1):
    ed[i,n] = m-i
  for j in range(n-1,-1,-1):
    ed[m,j] = n-j

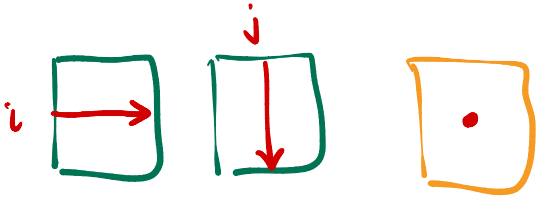
  for j in range(n-1,-1,-1):
    for i in range(m-1,-1,-1):
      if u[i] == v[j]:
        ed[i,j] = ed[i+1,j+1]
      else:
        ed[i,j] = 1 + min(ed[i+1,j+1],
                          ed[i,j+1],
                          ed[i+1,j])
  return(ed[0,0])
```

Complexity

- Again $O(mn)$, using dynamic programming or memoization
  - Fill a table of size $O(mn)$
  - Each table entry takes constant time to compute

Multiplying matrices

- Multiply matrices A, B
  - $AB[i,j] = \sum_{k=0}^{n-1} A[i,k]B[k,j]$



Multiplying matrices

- Multiply matrices A, B
  - $AB[i,j] = \sum_{k=0}^{n-1} A[i,k]B[k,j]$
- Dimensions must be compatible
  - $A : m \times n$, $B : n \times p$
  - $AB : m \times p$

Multiplying matrices

- Multiply matrices A, B
  - $AB[i,j] = \sum_{k=0}^{n-1} A[i,k]B[k,j]$
- Dimensions must be compatible
  - $A : m \times n$, $B : n \times p$
  - $AB : m \times p$
- Computing each entry in AB is $O(n)$
- Overall, computing AB is $O(mnp)$

$O(n^{\omega})$

– Square matrices $m = n = p$

$O(n^3)$

Multiplying matrices

- Multiply matrices A, B
  - $AB[i,j] = \sum_{k=0}^{n-1} A[i,k]B[k,j]$
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  - $A : m \times n$, $B : n \times p$
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- Matrix multiplication is associative
  - $ABC = (AB)C = A(BC)$

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- Matrix multiplication is associative
  - $ABC = (AB)C = A(BC)$
  - Bracketing does not change answer
  - ... but can affect the complexity!

Multiplying matrices

- Multiply matrices A, B
  - $AB[i,j] = \sum_{k=0}^{n-1} A[i,k]B[k,j]$
- Dimensions must be compatible
  - $A : m \times n$, $B : n \times p$
  - $AB : m \times p$
- Computing each entry in AB is $O(n)$
- Overall, computing AB is $O(mnp)$
- Matrix multiplication is associative
  - $ABC = (AB)C = A(BC)$
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  - . . . but can affect the complexity!

- Let $A : 1 \times 100$, $B : 100 \times 1$, $C : 1 \times 100$

Multiplying matrices

- Multiply matrices A, B
  - $AB[i,j] = \sum_{k=0}^{n-1} A[i,k]B[k,j]$
- Dimensions must be compatible
  - $A : m \times n$, $B : n \times p$
  - $AB : m \times p$
- Computing each entry in AB is $O(n)$
- Overall, computing AB is $O(mnp)$
- Matrix multiplication is associative
  - $ABC = (AB)C = A(BC)$
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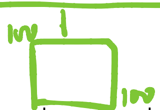
- Let $A : 1 \times 100$, $B : 100 \times 1$, $C : 1 \times 100$



- Computing $A(BC)$
  - $BC : 100 \times 100$, takes $100 \cdot 1 \cdot 100 = 10000$ steps to compute
  - $A(BC) : 1 \times 100$, takes $1 \cdot 100 \cdot 100 = 10000$ steps to compute

Multiplying matrices

- Multiply matrices A, B
  - $AB[i,j] = \sum_{k=0}^{n-1} A[i,k]B[k,j]$
- Dimensions must be compatible
  - $A : m \times n$, $B : n \times p$
  - $AB : m \times p$
- Computing each entry in AB is $O(n)$
- Overall, computing AB is $O(mnp)$
- Matrix multiplication is associative
  - $ABC = (AB)C = A(BC)$
  - Bracketing does not change answer
  - . . . but can affect the complexity!

- Let $A : 1 \times 100$, $B : 100 \times 1$, $C : 1 \times 100$
- Computing $A(BC)$
  - $BC : 100 \times 100$, takes $100 \cdot 1 \cdot 100 = 10000$ steps to compute
  - $A(BC) : 1 \times 100$, takes $1 \cdot 100 \cdot 100 = 10000$ steps to compute
- Computing $(AB)C$
  - $AB : 1 \times 1$, takes $1 \cdot 100 \cdot 1 = 100$ steps to compute
  - $(AB)C : 1 \times 100$, takes $1 \cdot 1 \cdot 100 = 100$ steps to compute

Multiplying matrices

- Multiply matrices A, B
  - $AB[i,j] = \sum_{k=0}^{n-1} A[i,k]B[k,j]$
- Dimensions must be compatible
  - $A : m \times n$, $B : n \times p$
  - $AB : m \times p$
- Computing each entry in AB is $O(n)$
- Overall, computing AB is $O(mnp)$
- Matrix multiplication is associative
  - $ABC = (AB)C = A(BC)$
  - Bracketing does not change answer
  - ... but can affect the complexity!

- Let $A : 1 \times 100$, $B : 100 \times 1$, $C : 1 \times 100$
- Computing $A(BC)$
  - $BC : 100 \times 100$, takes $100 \cdot 1 \cdot 100 = 10000$ steps to compute
  - $A(BC) : 1 \times 100$, takes $1 \cdot 100 \cdot 100 = 10000$ steps to compute
- Computing $(AB)C$
  - $AB : 1 \times 1$, takes $1 \cdot 100 \cdot 1 = 100$ steps to compute
  - $(AB)C : 1 \times 100$, takes $1 \cdot 1 \cdot 100 = 100$ steps to compute
- 20000 steps vs 200 steps!

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  - Product $M_0 \cdot M_1 \cdots M_{n-1}$ can be computed

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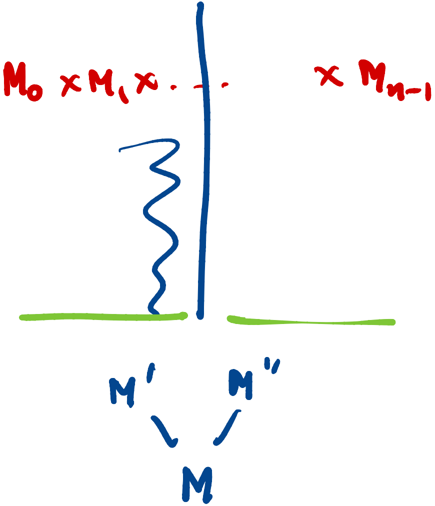
  - Product $M_0 \cdot M_1 \cdots M_{n-1}$ can be computed
- Find an optimal order to compute the product
  - Multiply two matrices at a time
  - Bracket the expression optimally

Inductive structure

- Final step combines two subproducts
  $(M_0 \cdot M_1 \cdots M_{k-1}) \cdot (M_k \cdot M_{k+1} \cdots M_{n-1})$
  for some $0 < k < n$



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  - Try all and choose the minimum!

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  - $M_0 \cdot M_1 \cdots M_{k-1}$ would decompose as $(M_0 \cdots M_{j-1}) \cdot (M_j \cdots M_{k-1})$
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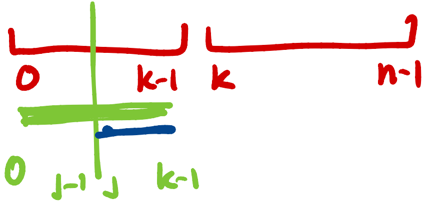

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  $\min_{j<\ell\leq k} [\underline{C(j, \ell-1) + C(\ell, k)} + \underline{r_j r_\ell c_k}]$

M_j | - - - M_k |

ℓ . | $j+1$ k |

Inductive structure

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- $C(j, k) =$
  $\min_{j<\ell\leq k} [C(j, \ell-1) + C(\ell, k) + r_j r_\ell c_k]$
- Base case: $C(j, j) = 0$ for $0 \leq j < n$

Subproblem dependency

- Compute $C(i,j)$, $0 \leq i,j < n$

	0	$\cdots$	i	$\cdots$	$\cdots$	j	$\cdots$	$n{-}1$
0								
$\cdots$								
i								
$\cdots$								
$\cdots$								
j								
$\cdots$								
$n{-}1$								

Subproblem dependency

- Compute $C(i,j)$, $0 \leq i,j < n$
  - Only for $i \leq j$
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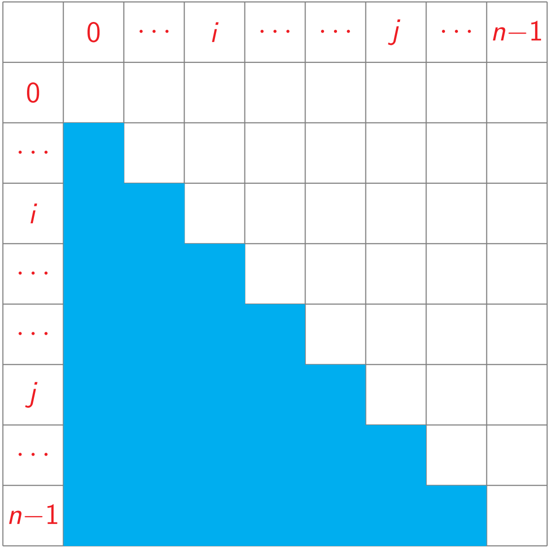

Subproblem dependency

- Compute $C(i,j)$, $0 \leq i,j < n$
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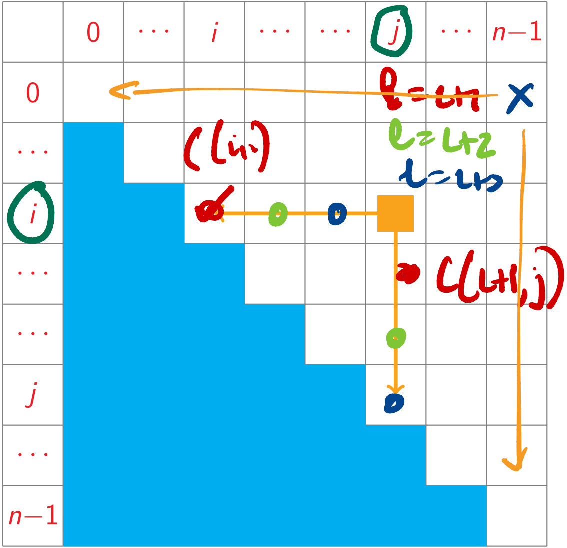
- $C(i,j)$ depends on $C(i,k-1)$, $C(k,j)$ for every $i < k \leq j$

$C(i,j)$

$C(i,\ell-1)$ $C(\ell,j)$

$i < \ell \leq j$



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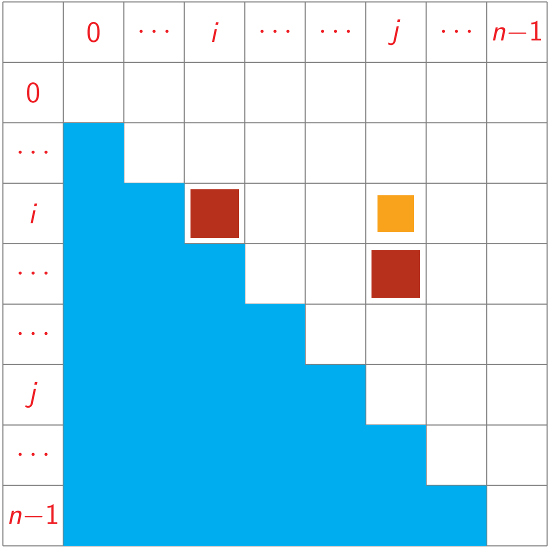

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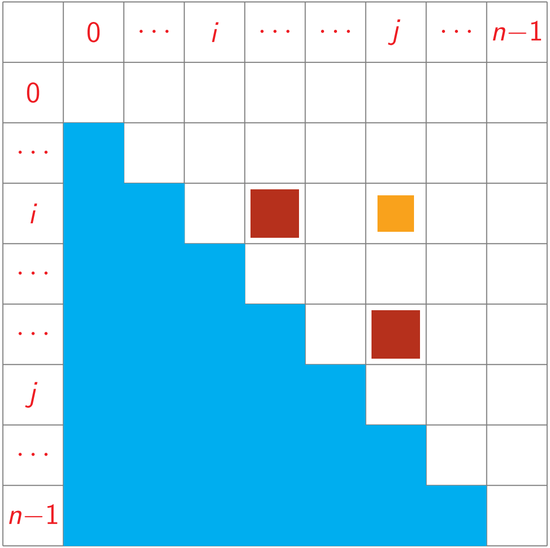

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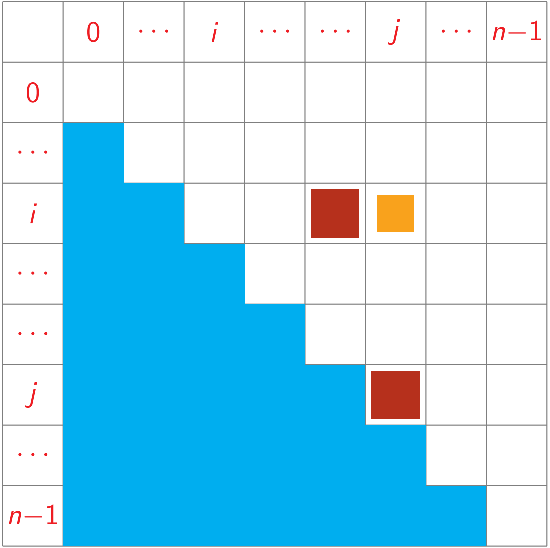

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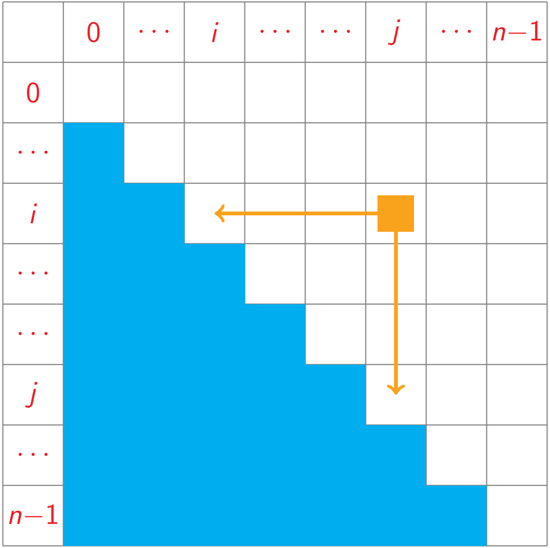

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- Diagonal entries are base case

	0	$\cdots$	i	$\cdots$	$\cdots$	j	$\cdots$	$n-1$
0	0							
$\cdots$		0						
i			0					
$\cdots$				0				
$\cdots$					0			
j						0		
$\cdots$							0	
$n-1$								0

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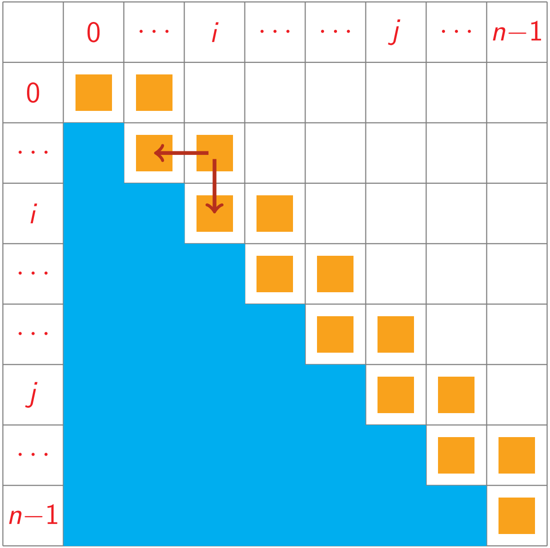

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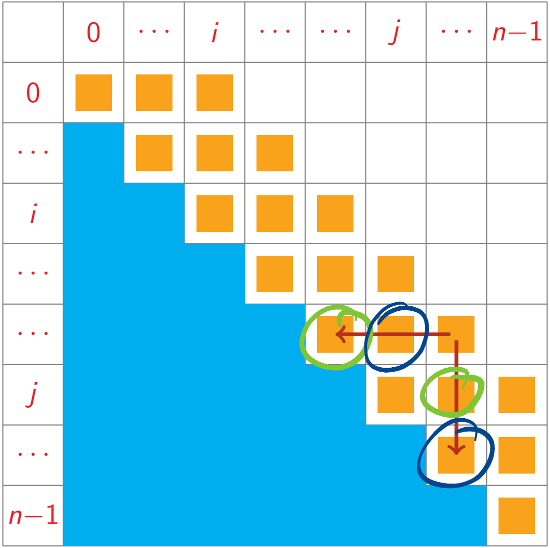

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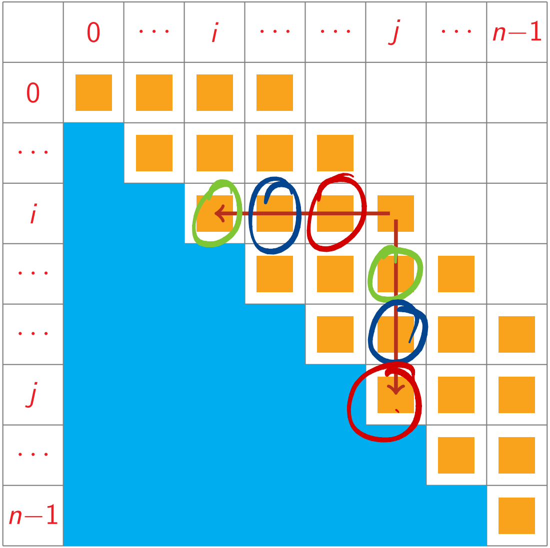

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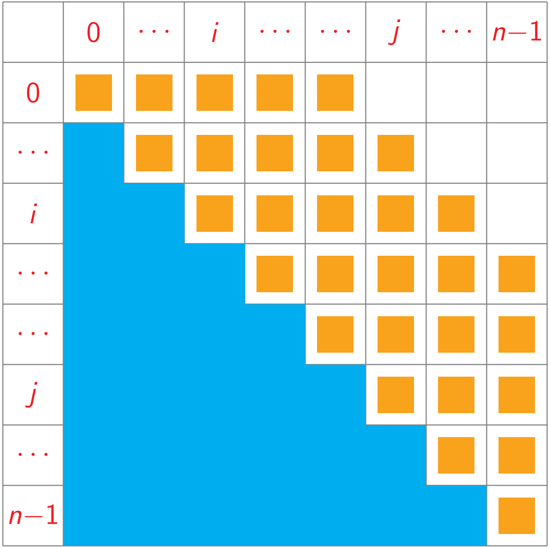

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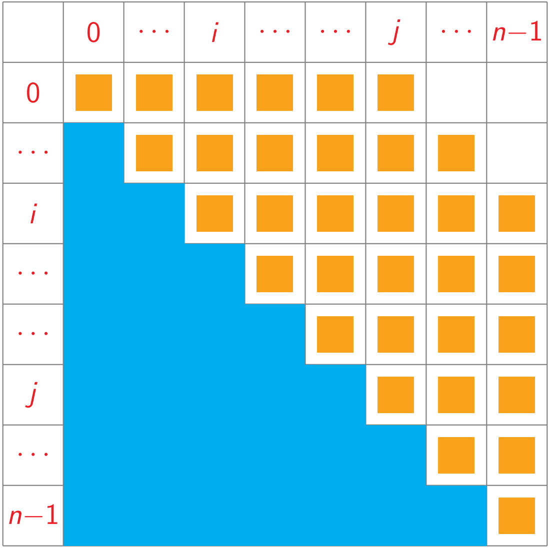

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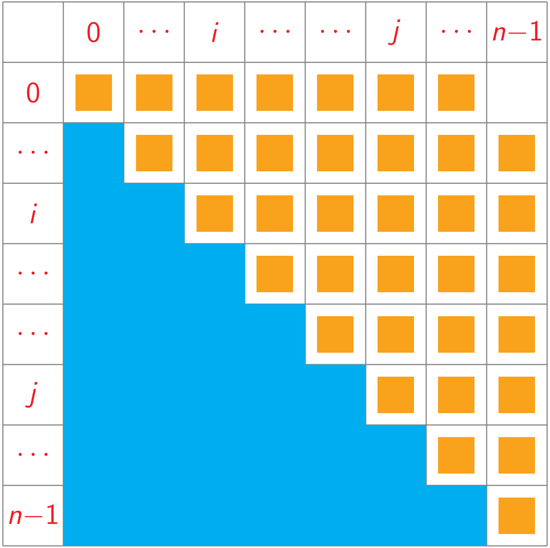

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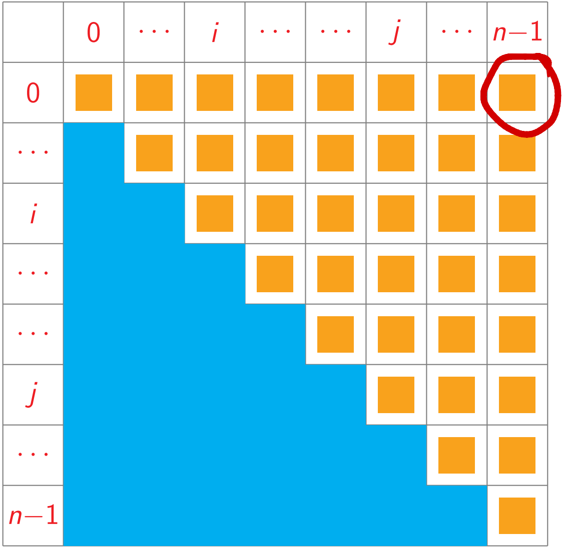
$C(0, n-1)$



Implementation

```
def C(dim):
  # dim: dimension matrix,
  #      entries are pairs (r_i,c_i)
  import numpy as np
  n = dim.shape[0]
  C = np.zeros((n,n))
  for i in range(n):
    C[i,i] = 0
  for diff in range(1,n):
    for i in range(0,n-diff):   || diagonal step
      j = i + diff
      C[i,j] = C[i,i] +
               C[i+1,j] +
               dim[i][0]*dim[i+1][0]*dim[j][1]
      for k in range(i+1,j+1):
        C[i,j] = min(C[i,j],
                     C[i,k-1] + C[k,j] +
                     dim[i][0]*dim[k][0]*dim[j][1])
  return(C[0,n-1])
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```

Complexity

- We have to fill a table of size $O(n^2)$
- Filling each entry takes $O(n)$
- Overall, $O(n^3)$